A New Oberbeck-Boussinesq approximation with density depending also on pressure and Bénard problem

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Introduction

The mathematical modeling of incompressible fluids has been object of considerable attention during the past decades. This is due to the fact that, although fully incompressible materials do not exist in nature, the incompressibility is a useful idealization of several materials which exhibit extreme resistance to volume change. In the context of isothermal mechanics, an ideal incompressible material is a medium that can only be deformed without any change in volume.

From an experimental point of view, an incompressible medium has no real existence but can be approximated as limit case of compressible one.

Starting from this observation much literature has been devoted, in particular by qualitative analysis and numerical methods, to search solutions of the incompressible case as the limit of solutions of the compressible ones as the Mach number tends to zero under certain assumptions on the initial data.
See, for example, in the isothermal case,

Klainerman S., Majda S. (1981) (1981,1982);
Lions P.L., Masmoudi N. (1998)
Desjardins B., Grenier E., Lions P.L., Masmoudi N. (1999).

When the process is not isothermal, the notion of incompressibility is not well defined and several possibilities arise.
In order to compare the solutions of compressible and incompressible media, it is suitable to choose the pressure p (in place of the density ρ) and the temperature T as thermodynamic independent variables. The other quantities, such as specific volume $V = 1/\rho$ and internal energy ε, being identified by constitutive equations in the form:

$$V \equiv V(p,T)\,, \quad \varepsilon \equiv \varepsilon(p,T).$$

Two parameters are important for a fluid: the *thermal expansion coefficient* α and the *compressibility factor* β defined by

$$\alpha = \frac{V_T}{V}, \qquad \beta = -\frac{V_p}{V}\,,$$

where the subscripts T and p denote the partial derivatives with respect to the variables T and p.

Experiments confirm that for those fluids usually handled as incompressible the volume changes a little with the temperature and in practice it remains unchanged by varying the pressure. For this reason, many authors name incompressible a material for which the specific volume does not vary unless the temperature varies (*i.e.*, $V \equiv V(T)$).

The first model of incompressibility was discussed by Müller [3], who chooses not only the specific volume V but all the thermodynamical functions independent of the pressure. Then, if also the internal energy ε depends just on the temperature T, in order to maintain the compatibility with the entropy principle, the functional form of V should be nothing else than a constant. As was previously stressed, this result disagrees with experimental and theoretical results, in particular for the Boussinesq approximation. Gouin, Muracchini and Ruggeri call this contradiction the *Müller paradox* [6].

A further and less restrictive model require that $V \equiv V(T)$ but the internal energy depends also on the pressure $\varepsilon \equiv \varepsilon(p, T)$ (see, for example Rajakopal et al., Gouin, Muracchini & Ruggeri) and as a consequence the Gibbs equation is not violated. In a recent paper [6], it is proved that for a pressure smaller than a critical value, the ideal medium of Müller can be recovered as a limit case.

Nonetheless, a weak point of $V \equiv V(T)$ as definition of incompressibility was first shown by Manacorda [8] who noticed that instability occurs in wave propagation. As observed by Gouin and Ruggeri in [7] the instabilities are due to the non-concavity of the chemical potential inducing imaginary sound velocity c. These authors gave a detailed analysis concerning the thermodynamic consistence and they proved

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Thermodynamical Consistence

Statement

Given the constitutive functions

$$V \equiv V(p,T), \qquad e \equiv e(T)$$

the thermodynamical restriction require that any compressible materials have an internal energy given by

$$\varepsilon(p,T) = e(T) + \int V\,dp - p\,V - T\int V_{T}\,dp,$$

with

$$V_p < -\frac{TV_T^2}{C_p} < 0, \quad \left(C_p = e'(T) - T\int V_{TT}dp > 0\right).$$

Statement

The previous inequality can be rewritten in terms of the thermal expansion and compressible coefficients:

$$\beta > \beta_{cr}, \qquad \beta_{cr} = \frac{T\alpha^2 V}{C_p}, \tag{1}$$

and the sound velocity assume the following expression:

$$c = \sqrt{\frac{V}{\beta - \beta_{cr}}}.$$

Therefore there is an lower bound for β*: when* $\beta \to \beta_{cr}$, $c \to \infty$.

Definition of Quasi Thermal Incompressibility (QTI)

As we say before the experiments confirms that for *incompressible fluids*, the volume little changes with temperature and does not practically change with pressure.
In the neighborhood of a reference state (p_0, T_0, V_0), we choose a small dimensionless parameter we note δ $(\delta << 1)$ such that:

$$\delta = \alpha_0 T_0, \tag{2}$$

and moreover, we assume that β_0 order is δ^2:

$$\beta_0 p_0 = O(\delta^2). \tag{3}$$

Definition

A compressible fluid satisfying thermodynamical conditions of previous section is called a Quasi Thermal Incompressible (QTI) fluid if there exist $\hat{V}(T)$ *and* $\hat{\varepsilon}(T)$ *such that:*

$$V(p,T) = \hat{V}(T) + O(\delta^2); \qquad \varepsilon(p,T) = \hat{\varepsilon}(T) + O(\delta^2) \tag{4}$$

with $\hat{V}'(T_0) = \delta\, V_0/T_0$.

The most significant case is the linear expansion of V near (p_0, T_0):

$$V = V_0\,(1 + \alpha\,(T - T_0) - \beta\,(p - p_0))\,, \quad e = C_p T, \tag{5}$$

in this case α, β and C_p are constants. It is important to notice that the condition $\beta > \beta_{cr}$ doesn't allow to neglect the pressure terms in the constitutive equation.

Field Equations

The balance law system of mass, momentum and energy with Navier-Stokes-Fourier constitutive equations can be written:

$$\begin{aligned} &\frac{d\rho}{dt} + \rho\nabla\cdot\mathbf{v} = 0 \\ &\rho\frac{d\mathbf{v}}{dt} = -\nabla p + (\lambda+\mu)\nabla(\nabla\cdot\mathbf{v}) + \mu\Delta\mathbf{v} + \rho\mathbf{g} \\ &\rho C_V\frac{dT}{dt} = -p\nabla\cdot\mathbf{v} + \lambda(\nabla\cdot\mathbf{v})^2 + 2\mu\mathbb{D}:\mathbb{D} + k\Delta T + \\ &\qquad\qquad + \left\{p - T\left(\frac{\partial p}{\partial T}\right)_\rho\right\}\nabla\cdot\mathbf{v}, \end{aligned} \tag{6}$$

where

$$C_V = \left(\frac{\partial\varepsilon}{\partial T}\right)_\rho$$

is the specific heat at constant volume.

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New Boussinesq approximation

Here are the assumptions of the O-B approximation with the variant due to constitutive equation (5) Passerini & Ruggeri (2014):

- $\nabla \cdot \boldsymbol{v} = 0$,
- $\rho = \rho_0(1 - \alpha_0 (T - T_0) + \beta_0 (p - p_0))$ in the weight force (with $\alpha, \beta > 0$),
- $\rho = \rho_0$ except in the buoyancy term,
- $\mathbb{D} : \mathbb{D} \approx 0$,

As a consequence of such assumptions, system (6) is simplified as follows

$$\nabla \cdot \mathbf{v} = 0$$

$$\rho_0 \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \mu \Delta \mathbf{v} + \rho_0 (1 - \alpha (T - T_0) + \beta (p - p_0)) \mathbf{g} \tag{7}$$

$$\rho C_V \left(\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T \right) = k \Delta T.$$

The Bénard problem

The most studied setting for Boussinesq approximation is the Bénard problem, consisting in a fluid between two horizontal parallel planes kept at constant distance h and constant different temperatures. If the horizontal plane is isotropic, the mathematical 2-D setting with z increasing with the height is physically meaningful: we mean that the convection rolls can be observed in the layer and that their invariance axes are normal to the random direction of the horizontal perturbation generating them. We can prescribe periodicity conditions in that arbitrary horizontal direction and call it x. Then -by thermostatically heating the boundaries so that $T(x,0) = T_d > T_u = T(x,h)$- several exact and observable results follow, that we wish to recover. We start our analysis by stressing (as evident) that the new model has the same symmetry features than the old one.

The basic solution

The rest state $\mathbf{v} = 0$ and the stratified temperature distribution given by

$$T(z) = T_d - \frac{\delta T}{h} z$$

(which solves $\Delta T = 0$ with $T(x,0) = T_d$ and $T(x,h) = T_d - \delta T$), can be inserted both in the classical O-B system, getting:

$$\frac{dp}{dz} = -\rho_0 g \left(1 + \alpha \frac{\delta T}{h} z\right) \longrightarrow p = p_0 - \rho_0 g \left(z + \alpha \frac{\delta T}{h} \frac{z^2}{2}\right) .$$

and in the new one

$$\frac{dp}{dz} + \rho_0 g \beta (p - p_0) = -\rho_0 g \left(1 + \alpha \frac{\delta T}{h} z\right) \longrightarrow$$

$$p(z) = p_0 + \frac{1}{\beta^2} \frac{\alpha \delta T}{\rho_0 g h} \left(1 - e^{-\rho_0 g \beta z}\right) - \frac{1}{\beta} \left(1 - e^{-\rho_0 g \beta z} + \frac{\alpha \delta T}{h} z\right) .$$

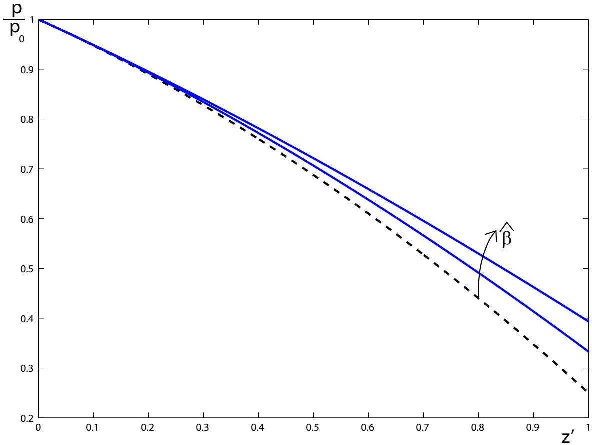

Figure: Behavior of dimensionless pressure p/p_0 versus $z' = z/h$ for increasing value of $\hat{\beta} = \rho_0 \beta g h$. We choose only for graphical convenience $\alpha \delta T = 1,\ \rho_0 g h / p_0 = 1/2$. The dashed curve is the classical parabolic profile corresponding to $\hat{\beta} = 0$, the other two curves are for $\hat{\beta} = 0.27$ and $\hat{\beta} = 0.5$ respectively.

New variables $P = p - p(z)$ and $\tau = T - T(z)$ are defined by difference with the basic solution, so achieving a system allowing for zero as solution:

$$\nabla \cdot \mathbf{v} = 0$$
$$\frac{1}{Pr}\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}\right) = -\nabla P - \rho_0 g \beta h P \mathbf{k} + \Delta \mathbf{v} + Ra \tau \mathbf{k}$$
$$\frac{\partial \tau}{\partial t} + \mathbf{v} \cdot \nabla \tau = \Delta \tau + v_z \ ,$$

with Robin boundary conditions $\nabla P \cdot \mathbf{k} + \rho_0 g \beta h P = 0$ for the pressure at the boundaries.

Dimensionless variables and parameters:

$$t' = \kappa/h^2 \qquad x' = x/h \qquad z' = z/h \qquad \tau' = \tau/(\delta T) \qquad P' = P\, h^2 \rho_0/\kappa\mu\,.$$

$$Pr = \mu/\rho_0\kappa \qquad Ra = \alpha\rho_0 h^3 g \delta T/\mu\kappa\,.$$

The saddle

For non zero solutions the system can not be solved separately for τ and $\mathbf{v}$ by simply projecting the second equation in a divergence free functional space

$$\Delta P + \rho_0 g \beta h \frac{\partial P}{\partial z} = -\frac{1}{Pr}(\nabla \mathbf{v}) : (\nabla \mathbf{v})^T + Ra \frac{\partial \tau}{\partial z}$$

$$\frac{1}{Pr}\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}\right) = -\nabla P - \rho_0 g \beta h P \mathbf{k} + \Delta \mathbf{v} + Ra \tau \mathbf{k}$$

$$\frac{\partial \tau}{\partial t} + \mathbf{v} \cdot \nabla \tau = \Delta \tau + v_z \ ,$$

The set of initial conditions giving a decaying solution for all Ra is $\tau(x, z, 0) = f(z)$

$$\tau(x, z, t) = 2 \sum_{n=1}^{\infty} \left(\int_0^1 f(s) \sin(n \pi s) ds \right) \sin(n \pi z) e^{-n^2 \pi^2 t}$$

$$P(x, z, t) = e^{-\rho_0 g \beta z} \Psi(z, t) \qquad with \ \frac{\partial \Psi}{\partial z} = Ra\, e^{\rho_0 g \beta z} \tau(x, z, t) \ ,$$

The linear system in 2-D

The streamfunction:

$$-\Phi_z = v^x \qquad\qquad \Phi_x = v^z\,.$$

By setting $P = e^{-\beta z}\Pi$ the Robin boundary condition becomes $\Pi_z = 0$.
Apply $\nabla\times$ to the momentum balance and linearize:

$$\begin{aligned}
&\Delta\Pi - \beta\Pi_z = Ra\, e^{\beta z}\tau_z \\
&-\frac{1}{Pr}\Delta\Phi_t + \Delta^2\Phi = Ra\,\tau_x - \hat{\beta}\,P_x \\
&\tau_t - \Delta\tau = \Phi_x \ ,
\end{aligned}$$

Periodicity condition in x, plus free surface condition for $\mathbf{v(x,z,t)}$ lead to ask $\Delta\Phi = 0$ for $z = 0$ and $z = 1$, the same condition as for τ.

$$\Psi_{mn}(x,z) = (d_1\cos 2\pi m x + d_2 \sin 2\pi m x)\sin \pi n z$$

$$\Phi_{mn}(x,z) = (d_1\cos 2\pi m x + d_2 \sin 2\pi m x)\cos \pi n z$$

and $\alpha_{mn} = (2\pi m)^2 + (\pi n)^2$ are the positive eigenvalues of $-\Delta$.

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The form of the solutions

$$\tau = \sum_{m,n}^{\infty} A_{mn} e^{\lambda_{mn} t} \Psi_{mn}(x,z) \qquad \Pi = \sum_{m,n} B_{mn} e^{\lambda_{mn} t} \Phi_{mn}(x,z)$$

$$\Delta\Phi = \sum_{m,n}^{\infty} C_{mn} e^{\lambda_{mn} t} \Psi_{mn}(x,z)\,.$$

By Galerkin method

$$B_{mk} = -\frac{Ra\,\hat{\beta}}{(2\pi m)^2 + (pik)^2} \sum_{n=0}^{\infty} A_{mn} \mathcal{M}_{nk}(\hat{\beta})$$

$$\mathcal{M}_{nk}(\hat{\beta}) := \pi n (e^{\hat{\beta}}(-1)^{k+n} - 1) \left(\frac{1}{\pi^2 (k+n)^2 + \hat{\beta}^2} + \frac{1}{\pi^2 (k-n)^2 + \hat{\beta}^2} \right),$$

$$C_{mk} = -A_{mk} \frac{\alpha_{mk}}{2\pi m} (\lambda_{mk} + \alpha_{mk})$$

By the scalar product with $\sin \pi n z$ and with the help of Werner formulas:

$$A_{mj}\left[+\frac{1}{Pr}\lambda_{mj}\frac{\alpha_{mj}}{2\pi m}(\lambda_{mj}+\alpha_{mj})+\frac{\alpha_{mj}^2}{2\pi m}(\lambda_{mj}+\alpha_{mj})\right]=$$

$$=RaA_{mj}2\pi m+Ra\hat{\beta}^2\sum_{n,k=0}^{\infty}\frac{2\pi m}{(2\pi m)^2+(\pi k)^2}A_{mn}\mathcal{M}_{nk}(\hat{\beta})\mathcal{N}_{kj}(\hat{\beta})$$

where

$$\mathcal{N}_{kj}(\hat{\beta}):=(1-e^{\hat{\beta}}(-1)^{k+j})\left(\frac{\pi(j+k)}{\pi^2(j+k)^2+\hat{\beta}^2}+\frac{\pi(j-k)}{\pi^2(j-k)^2+\hat{\beta}^2}\right),$$

This system of equations in the unknowns A_{mj}'s is verified by asking that each arbitrary (but fixed) entry holds one while all the others are zero.

$$\Downarrow$$

The eigenvalues λ_{mn}

$$\frac{1}{Pr}\lambda_{mj}\frac{\alpha_{mj}}{2\pi m}(\lambda_{mj}+\alpha_{mj})+\frac{\alpha_{mj}^2}{2\pi m}(\lambda_{mj}+\alpha_{mj})=$$
$$Ra2\pi m+Ra\hat{\beta}^2\sum_{k=0}^{\infty}\frac{2\pi m}{(2\pi m)^2+(\pi k)^2}\mathcal{M}_{jk}(\hat{\beta})\mathcal{N}_{kj}(\hat{\beta})$$

For all j, m, this condition is a second order algebraic equation for λ_{mj}:

$$\lambda_{mj}^2+\alpha_{mj}(1+Pr)\lambda+Pr\alpha_{mj}^2-PrRa\frac{4\pi^2m^2}{\alpha_{mj}}-PrRa\hat{\beta}^2\mathcal{F}_{mj}(\hat{\beta})=0$$

with

$$\mathcal{F}_{mj}(\hat{\beta})=\frac{4\pi^2m^2}{\alpha_{mj}}\sum_{k=0}^{\infty}\frac{1}{(2\pi m)^2+(\pi k)^2}\mathcal{M}_{jk}(\hat{\beta})\mathcal{N}_{kj}(\hat{\beta})\,.$$

For all couple of indices j, m, condition (3.8) is a second order algebraic equation for the eigenvalue λ_{mj}:

The solutions are:

$$\lambda_{mj} = -\frac{Pr+1}{2}\,\alpha_{mj} \pm \sqrt{\left(\frac{Pr+1}{2}\,\alpha_{mj}\right)^2 + Pr\left(Ra\frac{4\pi^2 m^2}{\alpha_{mj}} - \alpha_{mj}^2 - \hat{\beta}^2 Ra \mathcal{F}_{mj}(\hat{\beta})\right)} \tag{9}$$

They are continuous functions in the variable $\hat{\beta}$. In the limit as $\hat{\beta} \to 0$, they coincides with the eigenvalues of the old system and, consequently, the critical Rayleigh number for the onset of convection is in the limit the same. We recall how the old result can be achieved: the expression under square root is always positive if $\hat{\beta} = 0$ (it is the sum of two squares). Then one gets a positive eigenvalue if

$$Ra\frac{4\pi^2 m^2}{\alpha_{mj}} - \alpha_{mj}^2 > 0$$

and it is sufficient that this happens for one couple of indices (m, j), so that convection arises just if Ra is larger of the minimum value of $Ra(m,j) = \alpha^3_{mj}/4\pi^2 m^2$. It is clearly attained for $j = 1$ and then minimizing

$$Ra(m,1) = \frac{\alpha^3_{m1}}{4\pi^2 m^2} \tag{10}$$

as function of the continuous variable m. This is possible since we chose h as horizontal length just to simplify the notation: in fact, we could have chosen any arbitrary positive number to fix the periodicity cell; hence, the related spatial frequency is continuous.
By direct calculation, the minimum corresponds to

$$m = m^* = \frac{1}{2\sqrt{2}}$$

and the critical Rayleigh number is

$$Ra^* = \frac{\alpha^3_{m^*1}}{4\pi^2 m^{*2}} = \frac{27\pi^4}{4} \simeq 657.54.$$

Let us study the sign of the eigenvalues (9) of the new model. If $\hat{\beta} \neq 0$ then for each mode the eigenvalue becomes positive if

$$Ra\left(\frac{4\pi^2 m^2}{\alpha_{mj}} - \hat{\beta}^2 \mathcal{F}_{mj}(\hat{\beta})\right) - \alpha_{mj}^2 > 0\,.$$

We want to see how $\hat{\beta} \neq 0$ affects the critical value of Ra. In order to do this, we study the sign of $\mathcal{F}_{mj}(\hat{\beta})$ close to the couple of indices minimizing the classical case. Thus, we first evaluate $\mathcal{F}_{mj}$ for the pair which minimizes the classical problem $(m = m^*, j = 1)$ and for $\hat{\beta} = 0$. For small $\hat{\beta}$ the sign of $\mathcal{F}_{m1}(\hat{\beta})$ remains the same by a continuity argument. Using the power of Mathematica software we obtain:

$$\mathcal{F}_{m^*1}(0) = \frac{16}{3\pi^6}\sum_{k=0}^{\infty}\frac{(1+(-1)^k)^2(1+k^2)}{(k^2-1)^3(1+2k^2)} =$$
$$-\frac{2}{81\pi^6}\left(432 + 33\pi^2 + 16\sqrt{2}\coth\left(\frac{\pi}{2\sqrt{2}}\right)\right) \simeq -0.022 \tag{11}$$

Still by the same argument, if we take into account that the boundary of the range of $Ra(m,1)$ is continuously transformed in itself by varying $\hat{\beta}$, then for small $\hat{\beta}$ the new minimum (obviously belonging to the boundary of the range) is attained for m^{**} necessarily close to m^* (*close* depending on $\hat{\beta}$). Therefore, it follows that since $\hat{\beta}^2\mathcal{F}_{m1}(\beta)$ is negative in the neighborhood of $m = m^*, \hat{\beta} = 0$, it is negative close to $m = m^{**}, \hat{\beta} = 0$ too.

This finally proves that the new critical Rayleigh is smaller than the old one:

$$Ra^{**}(\hat{\beta}) = \inf_{m,j} \left\{ \frac{\alpha_{mj}^3}{4\pi^2 m^2 - \hat{\beta}^2 \alpha_{mj} \mathcal{F}_{mj}(\hat{\beta})} \right\} < Ra^*. \qquad (12)$$

The numerical results are in perfect agreement with the previous theoretical considerations. In Fig. 2, the function $Ra(m,1)$ given by (10) as function of m is plotted together with the new function:

$$Ra_{\hat{\beta}}(m,1) = \frac{\alpha_{m1}^3}{4\pi^2 m^2 - \hat{\beta}^2 \alpha_{m1} \mathcal{F}_{m1}(\hat{\beta})}.$$

As we expect the new function is smaller than the classic one whatever m is, and its minimum corresponds to a value of m very close to m^*. Numerical samples of the critical Ra are given in the Table1, while in Fig. 3 one can even appreciate the decay of Ra^{**} for increasing $\hat{\beta}$.

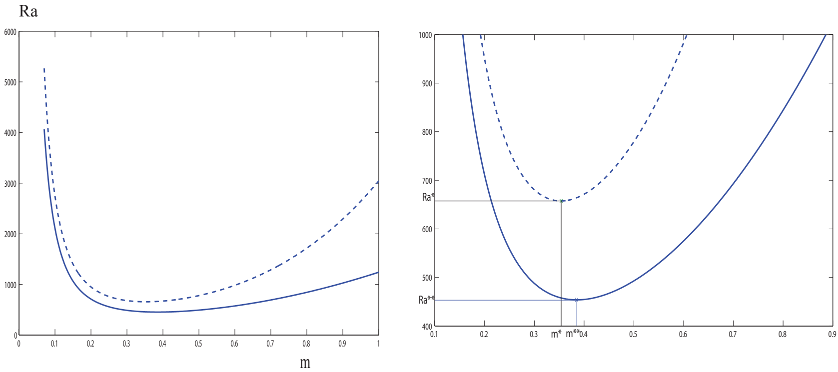

Figure: $Ra(m,1)$ (classic case with $\hat{\beta}=0$, dashed line) and present $Ra_{\hat{\beta}}(m,1)$ ($\hat{\beta}\neq 0$, continuous line) versus m. In the picture, $\hat{\beta}=0.12$ was chosen to emphasize the effect. On the right side: magnification close to the minimum values, which are identified as Ra^* and Ra^{**}; they respectively represent the classic and actual values of Ra.

Table 1 -	Critical values of Ra^{**} for different $\hat{\beta}$						
$\hat{\beta}$	$=$	0	0.02	0.04	0.06	0.08	0.1
Ra^{**}	$\simeq$	657.51	650.39	629.31	595.94	553.22	504.74

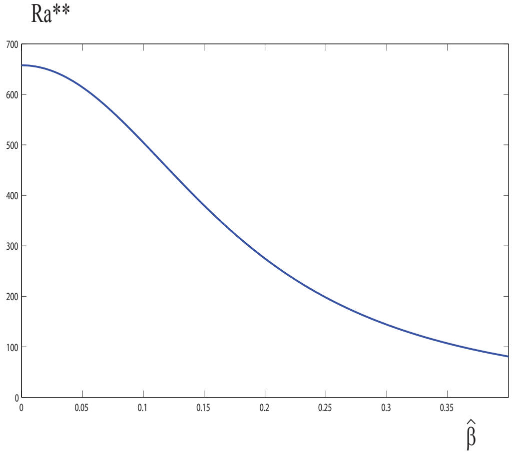

Figure: Decay of Ra^{**} for increasing $\hat{\beta}$.

Conclusions

In conclusion, we see that if one take into account the non zero compressibility in the model, trhough the pressure dependence in the density, the linear instability analysis changes. Necessary conditions for instability and non-linear stability analysis will appear soon in collaboration with S. Rionero.

In the case of Euler fluids, taking into account that the present model is hyperbolic as any compressible fluid it was possible to study shock waves and Riemann problem (Mentrelli & Ruggeri (2013) [9][10]).

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