

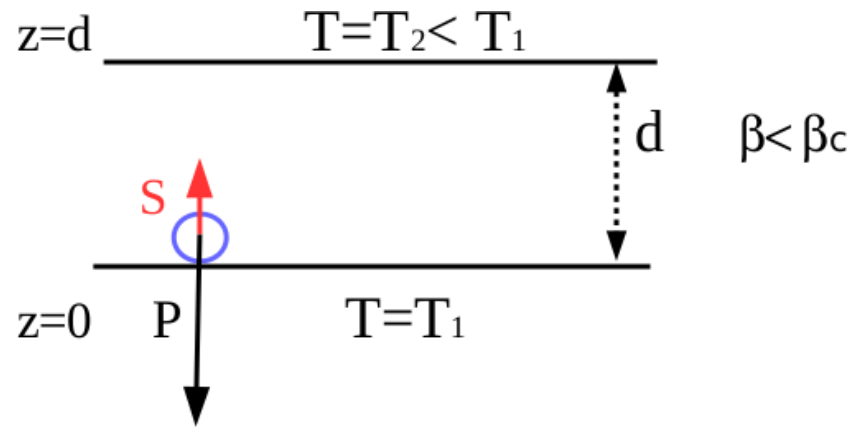
# Influence of depth-dependent Brinkman viscosity on the onset of convection in ternary porous layers

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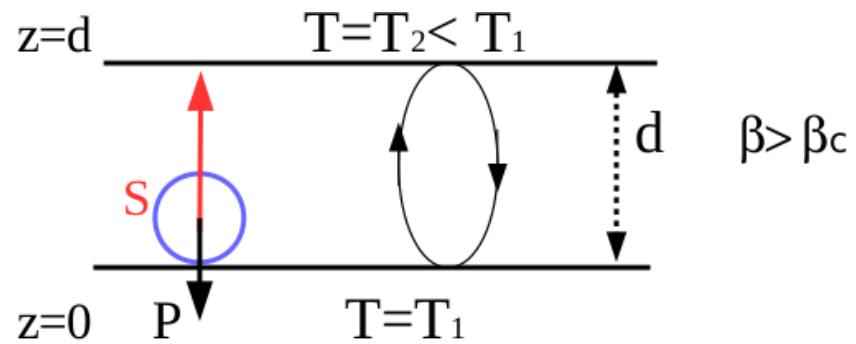
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$$\beta = \frac{T_1 - T_2}{d}, \quad \beta = \beta_c \Leftrightarrow \text{buoyancy force} = \text{weight}.$$

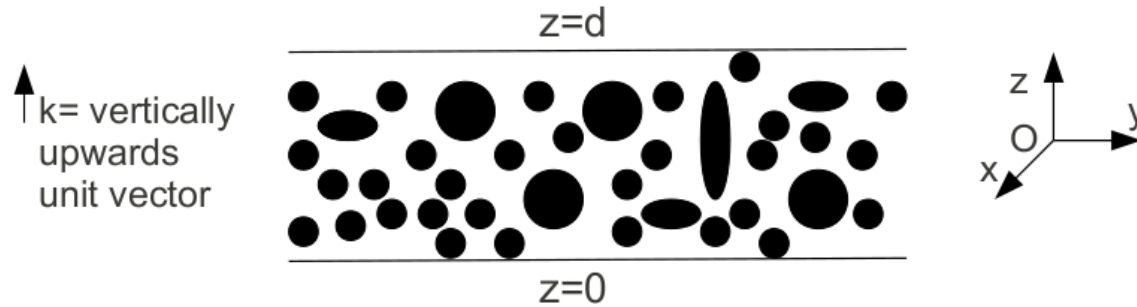


# Influence of depth-dependent Brinkman viscosity on the onset of convection in ternary porous layers

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# 1 Introduction



Because of its great relevance in geophysical phenomena and in planning new artificial porous materials, the onset of convection in porous layers with variable permeability and/or viscosities, in the past as nowadays, has attracted the attention of scientists { see [1]-[13], [28] and the references therein }. As concerns the geophysical phenomena, we confine ourselves to mention the increase in viscosity with depth in the earth's mantle [1]; the permeability changes due to mineral diagenesis in fractured crust [7]; the porosity changes due to subterranean movements, the increase in permeability and

porosity near solid wall {see [10]-[11] and references therein}. As concerns the stratification of porosity in artificial porous materials, we recall that for insulating purposes, the porosity has to be stratified in such a way to delay or inhibit heat transfer and hence in such a way to produce an high thermal critical Rayleigh number. On the contrary, when rapid heat transfer is requested (such as in cooling pipes used in modern devices), the porosity has to be stratified in such a way to produce low thermal critical Rayleigh numbers.

In the present paper, in the Darcy-Boussinesq-Brinkman scheme, the onset of convection in a porous horizontal layer  $L$  with depth-dependent permeability and viscosities - via the Auxiliary System Method [17]-[19] - is investigated<sup>1</sup>.

Section 2 is dedicated to some preliminaries and to the functional spaces in which the problem is embedded. In section 3 the main relation between the effective unknown fields is obtained. The linear instability is studied in the subsequent section 4. The nonlin-

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<sup>1</sup>In [28] the onset of convection in ternary porous layers with depth-dependent permeability and viscosity has been studied in the absence of Brinkman term.

ear stability is considered in section 5. In section 6, looking for symmetries and skew-symmetries hidden in the Darcy-Boussinesq-Brinkman model, for classes of values of the Prandtl numbers, a condition in closed form guaranteeing the global nonlinear stability, is furnished. Section 7 is devoted to the applications.

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## 2 Discussion

- i) The heat and mass transfer by convection in ternary Brinkman porous layers with depth-dependent permeability and viscosities, via the Auxiliary System Method, is studied<sup>2</sup>;*
- ii) an own non linear system is obtained for the evolution of each Fourier component of the perturbations;*
- iii) the linear instability captures completely the physics of the onset of convection since the absence of subcritical instabilities is shown together with the properties of linear stability to guarantee also the global non linear stability;*
- iv) the thermal critical Rayleigh number is obtained in closed form;*
- v) effects on the onset of convection in the earth's mantle, by virtue of depth-dependent permeability and viscosities, are*

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<sup>2</sup>Further applications of the auxiliary system method can be found in [16], [21]-[28]

*evaluated;*

*vi) how large pores should be stratified in artificial porous materials in order to inhibit or promote the onset of convection, is analyzed.*

### 3 Preliminaries

Let  $Ox_1x_2x_3$  be an orthogonal frame of reference with fundamental unit vectors  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$  ( $\mathbf{e}_3$  pointing vertically upwards).

We assume that two different chemical components (“salts”)  $S_\alpha$  ( $\alpha = 1, 2$ ), have dissolved in the fluid and have concentrations  $C_\alpha$  ( $\alpha = 1, 2$ ), respectively, and that the equation of state is

$$\rho = \rho_0 \left[ 1 - \alpha_*(T - T_0) + A_1(C_1 - \hat{C}_1) + A_2(C_2 - \hat{C}_2) \right],$$

where  $\rho_0, T_0, \hat{C}_\alpha$  ( $\alpha = 1, 2$ ), are reference values of the density, temperature and salt concentrations, while the constants  $\alpha_*, A_\alpha$  denote the thermal and solute  $S_\alpha$  expansion coefficients respectively ( $\alpha = 1, 2$ ). Combining Darcy’s Law with (thermal) energy and mass balance together with the Boussinesq approximation, we obtain the fundamental equations governing the isochoric motions (when the pores  $x_3 = z \in [0, 1]$  are in the layer **L**) sufficiently large to generate

the Brinkman viscosity) [14]-[16]

$$\left\{ \begin{array}{l} \nabla p = -\frac{\mu_1}{K} \mathbf{v} - \mathbf{g} \rho_0 [1 - \alpha_*(T - T_0) + A_1(C_1 - \hat{C}_1) + A_2(C_2 - \hat{C}_2)] + \\ + 2 \sum_{i,j}^{1-3} \frac{\partial}{\partial x_j} (\mu_2 D_{ij}) \mathbf{e}_i, \\ \nabla \cdot \mathbf{v} = 0, \\ T_t + \mathbf{v} \cdot \nabla T = k \Delta T, \\ C_{1t} + \mathbf{v} \cdot \nabla C_1 = k_1 \Delta C_1, \\ C_{2t} + \mathbf{v} \cdot \nabla C_2 = k_2 \Delta C_2, \end{array} \right. \quad (3.1)$$

where ( $i = 1, 2$ )

$p$  = pressure field,  $\mu_1 = f_1(x_3) \bar{\mu}_1$  = viscosity of the fluid,  
 $\mu_2 = f_2(x_3) \bar{\mu}_2$  = viscosity of the fluid in the porous layer,  
 $K = \bar{K} f_3(x_3)$  = permeability,  $\mathbf{v}$  = velocity,  $\mathbf{g}$  = gravity,  
 $k$  = thermal diffusivity,  $K_\alpha$  = diffusivity of the solute  $S_\alpha$ ,  
 $D_{ij} = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right),$

$\bar{K}$  = constant reference value of permeability;

$\bar{\mu}_i$  = constant reference value of viscosity  $\mu_i$ , ( $i = 1, 2$ ).

To (3.1) we append the boundary conditions

$$\begin{cases} T(0) = T_1, \quad T(d) = T_2, \\ C_\alpha(0) = C_{\alpha_l}, \quad C_\alpha(d) = C_{\alpha_u} \quad \alpha = 1, 2, \\ \mathbf{v} \cdot \mathbf{e}_3 = 0, \quad \text{at } z = 0, d, \end{cases} \quad (3.2)$$

with  $T_1, T_2, C_{\alpha_l}, C_{\alpha_u}$  ( $\alpha = 1, 2$ ), positive constants. The boundary value problem (3.1)-(3.2) admits the conduction solution  $(\tilde{\mathbf{v}}, \tilde{p}, \tilde{T}, \tilde{C}_\alpha)$  given by

$$\begin{cases} \tilde{\mathbf{v}} = 0, \quad \tilde{T} = T_1 - \beta z, \quad \beta = \frac{T_1 - T_2}{d}, \\ \tilde{C}_\alpha = C_{\alpha_l} - \frac{z(\delta C_\alpha)}{d}, \quad C_{\alpha_l} - C_{\alpha_u} = \delta C_\alpha, \\ \tilde{P} = p_0 + \rho_0 g z^2 \left[ -\frac{\alpha_* \beta}{2} + A_1 \frac{(\delta C_1)}{2d} + A_2 \frac{(\delta C_2)}{2d} \right] + \\ -\rho_0 g z \left[ 1 - \alpha_*(T_1 - T_0) + A_1(C_{1l} - \hat{C}_1) + A_2(C_{2l} - \hat{C}_2) \right], \end{cases} \quad (3.3)$$



where  $p_0$  is a constant. Setting

$$\mathbf{v} = \tilde{\mathbf{v}} + \mathbf{u}, \quad p = \tilde{P} + \Pi, \quad T = \tilde{T} + \theta, \quad C_\alpha = \tilde{C}_\alpha + \Phi_\alpha, \quad (3.4)$$

and introducing the scalings

$$\left\{ \begin{array}{l} t = t^* \frac{d^2}{k}, \quad \mathbf{u} = \mathbf{u}^* \frac{\bar{K}}{d}, \quad \Pi = \Pi^* \frac{\mu_1 k}{K}, \quad \mathbf{x} = \mathbf{x}^* d, \quad \theta = \theta^* T^\sharp, \\ \Phi_\alpha = (\Phi_\alpha)^* \Phi_\alpha^\sharp, \quad T^\sharp = \left( \frac{\bar{\mu}_1 k |\delta T|}{\alpha_* \rho_0 g \bar{K} d} \right)^{\frac{1}{2}}, \quad \Phi_\alpha^\sharp = \left( \frac{\bar{\mu}_1 k P_\alpha |\delta C_\alpha|}{A_\alpha \rho_0 g \bar{K} d} \right)^{\frac{1}{2}}, \\ R = \left( \frac{\alpha_* \rho_0 g \bar{K} d |\delta T|}{\mu_1 k} \right)^{\frac{1}{2}}, \quad R_\alpha = \left( \frac{A_\alpha \rho_0 g \bar{K} d P_\alpha |\delta C_\alpha|}{\bar{\mu}_1 k} \right)^{\frac{1}{2}}, \\ D_a = \frac{\bar{\mu}_2 k}{\bar{\mu}_1 d^2}, \quad D_{ij}^* = \frac{1}{2} \left( \frac{\partial u_i^*}{\partial x_j^*} + \frac{\partial u_j^*}{\partial x_i^*} \right), \\ \delta T = T_1 - T_2, \quad H = \text{sgn}(\delta T), \quad H_\alpha = \text{sgn}(\delta C_\alpha), \quad P_\alpha = \frac{k}{k_\alpha}, \end{array} \right. \quad (3.5)$$

since in the case at stake the layer is heated from below and salted from below by  $S_1$  and from above by  $S_2$ , it follows that  $H = H_1 = 1$ ,  $H_2 = -1$  and the equations governing the dimensionless perturbations  $\{\mathbf{u}^*, \Pi^*, \theta^*, (\Phi_\alpha)^*\}$ , omitting the stars, are

$$\left\{ \begin{array}{l} \nabla \Pi = -f_3(x_3)\mathbf{u} + (R\theta - R_1\Phi_1 - R_2\Phi_2)\mathbf{e}_3 + 2D_a \sum_{i,j}^{1-3} \frac{\partial}{\partial x_j} (gD_{ij})\mathbf{e}_i, \\ \nabla \cdot \mathbf{u} = 0, \\ \theta_t + \mathbf{u} \cdot \nabla \theta = R\mathbf{u} \cdot \mathbf{e}_3 + \Delta \theta, \\ P_1 \left( \frac{\partial \Phi_1}{\partial t} + \mathbf{u} \cdot \nabla \Phi_1 \right) = R_1\mathbf{u} \cdot \mathbf{e}_3 + \Delta \Phi_1, \\ P_2 \left( \frac{\partial \Phi_2}{\partial t} + \mathbf{u} \cdot \nabla \Phi_2 \right) = -R_2\mathbf{u} \cdot \mathbf{e}_3 + \Delta \Phi_2, \end{array} \right. \quad (3.6)$$

under the boundary conditions

$$\mathbf{u} \cdot \mathbf{e}_3 = \theta = \Phi_1 = \Phi_2 = 0 \text{ on } z = 0, 1. \quad (3.7)$$

In (3.5)-(3.6)  $R$  and  $R_\alpha$  are the thermal and salt Rayleigh numbers

respectively while  $P_\alpha$  are the salt Prandtl numbers and  $D_a$  is the Darcy number.

We assume, as usually done in stability problems in layers, that

- i) the perturbations  $(u, v, w, \theta, \Phi_1, \Phi_2)$  are periodic in the  $x$  and  $y$  directions, respectively of periods  $2\pi/a_x, 2\pi/a_y$ ;
- ii)  $\Omega = [0, 2\pi/a_x] \times [0, 2\pi/a_y] \times [0, 1]$  is the periodicity cell;
- iii)  $\mathbf{u}, \Phi_1, \Phi_2, \theta$  belong to  $W^{2,2}(\Omega)$  and are such that all their first derivatives and second spatial derivatives can be expanded in a Fourier series uniformly convergent in  $\Omega$

and denote by  $L_2^*(\Omega)$  the set of functions  $\Phi$  such that

- 1)  $\Phi : (\mathbf{x}, t) \in \Omega \times \mathbb{R}^+ \rightarrow \Phi(\mathbf{x}, t) \in \mathbb{R}, \Phi \in W^{2,2}(\Omega), \forall t \in \mathbb{R}^+, \Phi$  is periodic in the  $x$  and  $y$  directions of period  $\frac{2\pi}{a_x}, \frac{2\pi}{a_y}$  respectively and  $[\Phi]_{z=0} = [\Phi]_{z=1} = 0$ ;
- 2)  $\Phi$ , together with all the first derivatives and second spatial derivatives, can be expanded in a Fourier series absolutely uniformly

convergent in  $\Omega$ ,  $\forall t \in \mathbb{R}^+$ .

#### 4 The effective unknown fields

This Section is devoted to show that  $(\theta, \Phi_1, \Phi_2)$  are the effective unknown fields. Let us consider the b.v.p.

$$\left\{ \begin{array}{l} \nabla \Pi = -f(z)\mathbf{u} + (R\theta - R_1\Phi_1 - R_2\Phi_2)\mathbf{e}_3 + 2D_a \sum_{i,j}^{1-3} \frac{\partial}{\partial x_j} (gD_{ij})\mathbf{e}_i, \\ \nabla \cdot \mathbf{u} = 0, \\ \mathbf{u} \cdot \mathbf{e}_3 = \theta = \Phi_1 = \Phi_2 = 0, \quad z = 0, 1. \end{array} \right. \quad (4.1)$$

Since the set  $\{\sin n\pi z\}_{n \in \mathbb{N}}$  is a complete orthogonal system for  $L^2(0, 1)$ , then

$$\Phi \in \{w, \theta, \Phi_1, \Phi_2\} \rightarrow \Phi = \sum_{n=1}^{\infty} \Phi_n = \sum_{n=1}^{\infty} \tilde{\Phi}_n(x, y, t) \sin n\pi z. \quad (4.2)$$

By virtue of the periodicity in the  $x$  and  $y$  directions, one easily obtains

$$\Delta_1 \Phi_n = -a^2 \Phi_n, \quad \Delta \Phi_n = -\xi_n \Phi_n, \quad (4.3)$$

with

$$a^2 = a_x^2 + a_y^2, \quad \Delta_1 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad \xi_n = a^2 + n^2 \pi^2. \quad (4.4)$$

Further, setting

$$\zeta = (\text{rot} \mathbf{u}) \cdot \mathbf{e}_3 = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \quad (4.5)$$

from (4.1)<sub>2</sub> one obtains

$$\Delta_1 u = -\frac{\partial^2 w}{\partial x \partial z} - \frac{\partial \zeta}{\partial y}, \quad \Delta_1 v = -\frac{\partial^2 w}{\partial y \partial z} + \frac{\partial \zeta}{\partial x}. \quad (4.6)$$

Since

$$\begin{cases} \left[ \frac{g(x_3)}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right]_j = \frac{1}{2} \frac{\partial g}{\partial x_j} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \frac{1}{2} g \left( \frac{\partial^2 u_i}{\partial x_j^2} + \frac{\partial}{\partial x_i} \frac{\partial u_i}{\partial x_j} \right), \\ \frac{\partial}{\partial x_i} \frac{\partial u_j}{\partial x_j} = \frac{\partial}{\partial x_i} (\nabla \cdot \mathbf{u}) = 0, \quad \frac{\partial g}{\partial x_j} = 0, \quad j = 1, 2, \end{cases} \quad (4.7)$$

it follows that

$$2(g(z)D_{ij})_j \mathbf{e}_i = g'(z) \left( \frac{\partial \mathbf{u}}{\partial z} + \nabla w \right) + g(z) \Delta \mathbf{u} \quad (4.8)$$

and (4.1) becomes

$$\nabla \Pi = -f(z)\mathbf{u} + \left( R\theta - \sum_{\alpha=1}^2 R_{\alpha}\Phi_{\alpha} \right) \mathbf{e}_3 + D_a [g'(z)(\mathbf{u}_z + \nabla w) + g\Delta \mathbf{u}]. \quad (4.9)$$

Setting

$$\mathbf{F} = q(z)\mathbf{U}, \quad \text{with } q \in C^2(0,1), \quad \nabla \cdot \mathbf{U} = 0, \quad (4.10)$$

one easily obtains

$$\mathbf{e}_3 \cdot \nabla \times \nabla \times \mathbf{F} = -q' \frac{\partial U_3}{\partial z} - q\Delta U_3. \quad (4.11)$$

Applying (4.11) with  $\mathbf{F}$  given respectively by  $f(z)\mathbf{u}$ ,  $g'\mathbf{u}_z$ ,  $g\Delta \mathbf{u}$ , one obtains

$$\begin{cases} \mathbf{e}_3 \cdot \nabla \times \nabla \times (f\mathbf{u}) = -f' \frac{\partial w}{\partial z} - f\Delta w, \\ \mathbf{e}_3 \cdot \nabla \times \nabla \times (g'\mathbf{u}_z) = -g''w_{zz} - g'\Delta w_z, \\ \mathbf{e}_3 \cdot \nabla \times \nabla \times (g\Delta \mathbf{u}) = -g'\Delta w_z - g\Delta \Delta w. \end{cases} \quad (4.12)$$

Since

$$\begin{cases} \mathbf{e}_3 \cdot \nabla \times \nabla \times (g' \nabla w) = g'' \Delta_1 w, \\ \mathbf{e}_3 \cdot \nabla \times \nabla \times \left( R\theta - \sum_{\alpha=1}^2 R_\alpha \Phi_\alpha \right) = -\Delta_1 \left( R\theta - \sum_{\alpha=1}^2 R_\alpha \Phi_\alpha \right), \end{cases} \quad (4.13)$$

the third component of the double curl of (4.1), is given by

$$A + B = -\Delta_1 \left( R\theta - \sum_{\alpha=1}^2 R_\alpha \Phi_\alpha \right), \quad (4.14)$$

with

$$A = -(f' w_z + f \Delta w), \quad B = g \Delta \Delta w + 2g' \Delta w_z + g'' (w_{zz} - \Delta_1 w). \quad (4.15)$$

For  $(w = w_n, \theta = \theta_n, \Phi_\alpha = \Phi_{\alpha n})$  one obtains

$$(A_{1n} + B_{1n}) \tilde{w}_n = a^2 \left( R\theta_n - \sum_{\alpha=1}^2 R_\alpha \Phi_{\alpha n} \right), \quad (4.16)$$



with

$$\begin{cases} A_{1n} = - \left( f' \frac{\partial}{\partial z} + f \Delta \right) \sin n\pi z, \\ B_{1n} = g \Delta \Delta + 2g' \Delta \frac{\partial}{\partial z} + g'' \left( \frac{\partial^2}{\partial z^2} - \Delta_1 \right) \sin n\pi z, \end{cases} \quad (4.17)$$

(4.16), multiplied by  $\sin n\pi z$  and integrated on  $(0, 1)$  gives

$$(A_n^* + B_n^*) \tilde{w}_n = \frac{a^2}{2} \left( R \tilde{\theta}_n - \sum_{\alpha=1}^2 R_\alpha \tilde{\Phi}_{\alpha n} \right), \quad (4.18)$$

with

$$\begin{cases} A_n^* = - \int_0^1 \left( \frac{n\pi}{2} f' \sin 2n\pi z + \xi_n f \sin^2 n\pi z \right) dz, \\ B_n^* = \int_0^1 \left\{ [g \xi_n^2 + g'' (\xi_n + 2a^2)] \sin^2 n\pi z - g' \xi_n \sin 2n\pi z \right\} dz, \end{cases} \quad (4.19)$$

i.e.

$$\tilde{w}_n = \tilde{A}_n \left( R\tilde{\theta}_n - \sum_{\alpha=1}^2 R_\alpha \tilde{\Phi}_{\alpha n} \right), \quad (4.20)$$

with

$$\tilde{A}_n = \frac{a^2}{2(A_n^* + B_n^*)}. \quad (4.21)$$

## 5 Linear instability

Setting

$$\mathcal{L}_n = \begin{pmatrix} a_{1n} & a_{2n} & a_{3n} \\ b_{1n} & b_{2n} & b_{3n} \\ c_{1n} & c_{2n} & c_{3n} \end{pmatrix}, \quad (5.1)$$

$$\left\{ \begin{array}{lll} a_{1n} = R^2 \tilde{\mathcal{A}}_n - \xi_n, & a_{2n} = -RR_1 \tilde{\mathcal{A}}_n, & a_{3n} = -RR_2 \tilde{\mathcal{A}}_n, \\ b_{1n} = \frac{RR_1}{P_1} \tilde{\mathcal{A}}_n, & b_{2n} = -\frac{R_1^2 \tilde{\mathcal{A}}_n + \xi_n}{P_1}, & b_{3n} = -\frac{RR_1}{P_1} \tilde{\mathcal{A}}_n, \\ c_{1n} = -\frac{RR_2}{P_2} \tilde{\mathcal{A}}_n, & c_{2n} = \frac{R_1 R_2}{P_2} \tilde{\mathcal{A}}_n, & c_{3n} = \frac{R_2^2 \tilde{\mathcal{A}}_n - \xi_n}{P_2}, \end{array} \right. \quad (5.2)$$

one easily obtains that

$$\frac{\partial}{\partial t} \sum_{n=1}^{\infty} \begin{pmatrix} \theta_n \\ \Phi_{1n} \\ \Phi_{2n} \end{pmatrix} = \sum_{n=1}^{\infty} \mathcal{L}_n \begin{pmatrix} \theta_n \\ \Phi_{1n} \\ \Phi_{2n} \end{pmatrix} - \mathbf{u} \cdot \sum_{n=1}^{\infty} \begin{pmatrix} \nabla \theta_n \\ \nabla \Phi_{1n} \\ \nabla \Phi_{2n} \end{pmatrix}. \quad (5.3)$$

Linearizing it follows that

$$\frac{\partial}{\partial t} \begin{pmatrix} \tilde{\theta}_n \\ \tilde{\Phi}_{1n} \\ \tilde{\Phi}_{2n} \end{pmatrix} = \mathcal{L}_n \begin{pmatrix} \tilde{\theta}_n \\ \tilde{\Phi}_{1n} \\ \tilde{\Phi}_{2n} \end{pmatrix}, \quad \forall n \in \{1, 2, \dots\}. \quad (5.4)$$

Denoting by  $(\lambda_{1n}, \lambda_{2n}, \lambda_{3n})$  the eigenvalues of  $\mathcal{L}_n$ , the spectral equation of  $\mathcal{L}_n$  is given by

$$\lambda_{in}^3 - \mathbf{I}_{1n}\lambda_{in}^2 + \mathbf{I}_{2n}\lambda_{in} - \mathbf{I}_{3n} = 0, \quad n \in \{1, 2, \dots\}, \quad (5.5)$$

with  $\mathbf{I}_{1n}, \mathbf{I}_{2n}, \mathbf{I}_{3n}$  characteristic values (invariants) of  $\mathcal{L}_n$  [20]-[21], given by

$$\left\{ \begin{array}{l} \mathbf{I}_{1n} = a_{1n} + b_{2n} + c_{3n} = \sum_{\alpha=1}^3 \lambda_{\alpha n}, \quad \mathbf{I}_{3n} = \det \mathcal{L}_n = \lambda_{1n} \lambda_{2n} \lambda_{3n}, \\ \mathbf{I}_{2n} = \begin{vmatrix} a_{1n} & a_{2n} \\ b_{1n} & b_{2n} \end{vmatrix} + \begin{vmatrix} a_{1n} & a_{3n} \\ c_{1n} & c_{3n} \end{vmatrix} + \begin{vmatrix} b_{2n} & b_{3n} \\ c_{2n} & c_{3n} \end{vmatrix} = \lambda_{1n}(\lambda_{2n} + \lambda_{3n}) + \lambda_{2n}\lambda_{3n} \end{array} \right. \quad (5.6)$$

and one easily obtains

$$\begin{cases} \mathcal{I}_{1n} = \left\{ R^2 - \left[ \frac{R_1^2}{P_1} - \frac{R_2^2}{P_2} + \frac{\xi_n}{\tilde{A}_n} \left( 1 + \frac{1}{P_1} + \frac{1}{P_2} \right) \right] \right\} \tilde{\mathcal{A}}_n, \\ \mathcal{I}_{2n} = \frac{P_1 + P_2}{P_1 P_2} \left[ \frac{1 + P_1 + P_2}{P_1 + P_2} \frac{\xi_n}{\tilde{A}_n} + \frac{1 + P_2}{P_1 + P_2} R_1^2 - \frac{1 + P_1}{P_1 + P_2} R_2^2 - R^2 \right] \xi_n \tilde{\mathcal{A}}_n, \\ \mathcal{I}_{3n} = \frac{1}{P_1 P_2} \left[ R^2 - \left( R_1^2 - R_2^2 + \frac{\xi_n}{\tilde{A}_n} \right) \right] \tilde{\mathcal{A}}_n \xi_n^2. \end{cases} \quad (5.7)$$

By virtue of the Routh-Hurwitz conditions on the sign of the real parts of the eigenvalues of  $\mathcal{L}_n$  ([20], pp. 111-114), the following results hold:

*i) the conditions,  $\forall (a^2, n) \in \mathbb{R}^+ \times \mathbb{N}$ ,*

$$\mathcal{I}_{1n} < 0, \quad \mathcal{I}_{2n} > 0, \quad \mathcal{I}_{3n} < 0, \quad (5.8)$$

*are necessary for guaranteeing the linear asymptotic stability;*

ii) if and only if,  $\forall (a^2, n) \in \mathbb{R}^+ \times \mathbb{N}$ ,

$$I_{1n} < 0, \quad I_{3n} < 0, \quad I_{1n}I_{2n} - I_{3n} < 0, \quad (5.9)$$

the thermal conduction solution is asymptotically linearly stable.

Setting

$$H^* = \min_{(a^2, n) \in \mathbb{R}^+ \times \mathbb{N}} H_n, \quad H_n = \frac{\xi_n}{\tilde{\mathcal{A}}_n}, \quad (5.10)$$

$$\begin{cases} R_{C_1} = \frac{R_1^2}{P_1} - \frac{R_2^2}{P_2} + \left(1 + \frac{1}{P_1} + \frac{1}{P_2}\right) H^*, & R_{C_3} = R_1^2 - R_2^2 + H^*, \\ R_{C_2} = \frac{1 + P_2}{P_1 + P_2} R_1^2 - \frac{1 + P_1}{P_1 + P_2} R_2^2 + \left(1 + \frac{1}{P_1 + P_2}\right) H^*, \end{cases} \quad (5.11)$$

in view of i) and ii) one obtains that

iii) the conditions

$$R^2 < R_{C_\alpha}, \quad \alpha = 1, 2, 3, \quad (5.12)$$

are necessary for inhibiting the onset of convection;  
 iv) if and only if

$$\begin{cases} R^2 < \min(R_{C_1}, R_{C_2}), \\ (R_{C_1} - R^2)(R_{C_2} - R^2) > \frac{H^*}{P_1 + P_2}(R_{C_3} - R^2), \end{cases} \quad (5.13)$$

convection cannot occur and the thermal conduction solution is asymptotically linearly stable.

Since (5.12) and (5.13)<sub>1</sub> are easily obtained, we confine ourselves to (5.13)<sub>2</sub>. But it is easily verified that  $\mathbf{I}_{2n} > \frac{\mathbf{I}_{3n}}{\mathbf{I}_{1n}}$  is equivalent to

$$\begin{aligned} & \left(1 + \frac{1}{P_1 + P_2}\right) \frac{\xi_n}{\tilde{\mathcal{A}}_n} + \frac{1 + P_2}{P_1 + P_2} R_1^2 - \frac{1 + P_1}{P_1 + P_2} R_2^2 > \\ & \frac{\xi_n / \tilde{\mathcal{A}}_n}{P_1 + P_2} \frac{(R_1^2 - R_2^2 + \xi_n / \tilde{\mathcal{A}}_n) - R^2}{\frac{R_1^2}{P_1} - \frac{R_2^2}{P_2} + \left(1 + \frac{1}{P_1} + \frac{1}{P_2}\right) \frac{\xi_n}{\tilde{\mathcal{A}}_n} - R^2}, \end{aligned} \quad (5.14)$$

which implies (5.13)<sub>2</sub>.

## 6 Nonlinear stability via the Auxiliary System Method

Setting

$$\left\{ \begin{array}{l} \mathbf{X} = \begin{pmatrix} \theta \\ \Phi_1 \\ \Phi_2 \end{pmatrix}, \quad \mathbf{F} = -\mathbf{u} \cdot \begin{pmatrix} \nabla \theta \\ \nabla \Phi_1 \\ \nabla \Phi_2 \end{pmatrix} = -\mathbf{u} \cdot \nabla \mathbf{X}, \\ \mathbf{X} = \sum_{n=1}^{\infty} \mathbf{X}_n, \quad \mathbf{X}_n = \begin{pmatrix} \theta_n \\ \Phi_{1n} \\ \Phi_{2n} \end{pmatrix}, \quad \mathbf{F}_n = -\mathbf{u} \cdot \nabla \mathbf{X}_n, \quad \mathbf{F} = \sum_{n=1}^{\infty} \mathbf{F}_n, \end{array} \right. \quad (6.1)$$

(5.4) become

$$\left\{ \begin{array}{l} \frac{\partial \mathbf{X}}{\partial t} = \sum_{n=1}^{\infty} (\mathcal{L}_n \mathbf{X}_n + \mathbf{F}_n), \\ (\mathbf{X})_{t=0} = \mathbf{X}^{(0)} = \sum_{n=1}^{\infty} \mathbf{X}_n^{(0)}, \quad \mathbf{X}_n = 0, \quad \text{on } z = 0, 1, \end{array} \right. \quad (6.2)$$



with  $\mathbf{X}_n^{(0)}$  assigned. Following [17]-[19], we call *auxiliary evolution system of the  $n$ th-Fourier component* of the perturbation  $\mathbf{X}$ , associated to the velocity  $\mathbf{u}$  given by (6.2), the system

$$\begin{cases} \frac{\partial}{\partial t} \tilde{\mathbf{X}}_n = \mathcal{L}_n \tilde{\mathbf{X}}_n + \tilde{\mathbf{F}}_n, \\ \tilde{\mathbf{X}}_n = 0, \quad \text{on } z = 0, 1, \quad [\tilde{\mathbf{X}}_n]_{t=0} = \mathbf{X}_n^{(0)}, \end{cases} \quad (6.3)$$

where  $\tilde{\mathbf{F}}_n$  and  $\tilde{\mathbf{X}}_n$  are given by

$$\tilde{\mathbf{F}}_n = -\mathbf{u} \cdot \nabla \tilde{\mathbf{X}}_n, \quad \tilde{\mathbf{X}}_n = \begin{pmatrix} \varphi_n \\ \varphi_{1n} \\ \varphi_{2n} \end{pmatrix}. \quad (6.4)$$

**Theorem 6.1** *Let  $\theta, \Phi_1, \Phi_2, \varphi_n, \varphi_{1n}, \varphi_{2n} \in L_2^*(\Omega)$  and let  $(\mathbf{u}, \theta, \Phi_1, \Phi_2)$  be solution of (6.2) and  $(\varphi_n, \varphi_{1n}, \varphi_{2n})$  be solution,  $\forall n \in \mathbb{N}$ , of*

*(6.3). Then the series  $\sum_{n=1}^{\infty} \varphi_n, \sum_{n=1}^{\infty} \varphi_{in}, (i = 1, 2)$  are a.e. con-*

vergent in  $\Omega$  and it follows that

$$\sum_{n=1}^{\infty} \varphi_n = \theta, \quad \sum_{n=1}^{\infty} \varphi_{in} = \Phi_i, \quad (i = 1, 2). \quad (6.5)$$

**Proof.** Setting:  $S_m = \sum_{n=1}^m \varphi_n$ ,  $S_{im} = \sum_{n=1}^m \varphi_{in}$ , ( $i = 1, 2$ ), the following i.b.v.p. holds

$$\frac{\partial}{\partial t} \begin{pmatrix} S_m \\ S_{1m} \\ S_{2m} \end{pmatrix} = \sum_{n=1}^m \mathcal{L}_n \begin{pmatrix} \varphi_n \\ \varphi_{1n} \\ \varphi_{2n} \end{pmatrix} - \begin{pmatrix} \mathbf{u} \cdot \nabla S_m \\ \mathbf{u} \cdot \nabla S_{1m} \\ \mathbf{u} \cdot \nabla S_{2m} \end{pmatrix}, \quad (6.6)$$

$$\begin{cases} (S_m)_{t=0} = \sum_{n=1}^m \theta_n^{(0)}, \quad (S_{im})_{t=0} = \sum_{n=1}^m \Phi_{in}^{(0)}, \quad (i = 1, 2), \\ S_m = S_{im} = \varphi_n = \varphi_{in} = 0, \quad z = 0, 1, \quad i = 1, 2 \quad n = 1, \dots, m. \end{cases} \quad (6.7)$$

Setting

$$\Psi_n = \begin{cases} \theta_n - \varphi_n, & \text{for } n = 1, 2, \dots, m, \\ \theta_n, & \text{for } n > m, \end{cases}, \quad \Psi = \sum_{n=1}^{\infty} \Psi_n \quad (6.8)$$

$$\Psi_{in} = \begin{cases} \Phi_{in} - \varphi_{in}, & \text{for } n = 1, 2, \dots, m, \\ \Phi_{in}, & \text{for } n > m, \end{cases}, \quad \Psi_i = \sum_{n=1}^{\infty} \Psi_{in} \quad (6.9)$$

by virtue of (6.3) and (6.6)-(6.7), one obtains,

$$\frac{\partial}{\partial t} \begin{pmatrix} \Psi \\ \Psi_1 \\ \Psi_2 \end{pmatrix} = \sum_{n=1}^{\infty} \mathcal{L}_n \begin{pmatrix} \Psi_n \\ \Psi_{1n} \\ \Psi_{2n} \end{pmatrix} - \begin{pmatrix} \mathbf{u} \cdot \nabla \Psi \\ \mathbf{u} \cdot \nabla \Psi_1 \\ \mathbf{u} \cdot \nabla \Psi_2 \end{pmatrix}, \quad (6.10)$$

under the i.b.c. ( $\alpha = 1, 2$ )

$$(\Psi)_{t=0} = \sum_{n=m+1}^{\infty} \theta_n^{(0)}, \quad (\Psi_i)_{t=0} = \sum_{n=m+1}^{\infty} \Psi_{in}^{(0)}; \quad \Psi = \Psi_\alpha = 0, \quad z = 0, 1. \quad (6.11)$$

Since  $\lim_{m \rightarrow \infty} \sum_{n=m+1}^{\infty} \theta_n^{(0)} = \lim_{m \rightarrow \infty} \sum_{n=m+1}^{\infty} \Phi_{in}^{(0)} = 0$  and (6.10) under zero i.b.v. admits only the null solution, it follows that

$$\lim_{m \rightarrow \infty} (\theta - S_m) = \lim_{m \rightarrow \infty} (\Phi_i - S_{im}) = 0, \quad (i = 1, 2). \quad (6.12)$$

**Theorem 6.2** *Let (5.13) hold. Then the zero solution of (6.2) is globally asymptotically stable, i.e. the thermal conduction solution is linearly stable and non linearly globally asymptotically stable with respect to the  $L^2(\Omega)$ -norm.*

**Proof.** The proof can be obtained following step by step the proof of theorem 7.1 given in [17] in the absence of depth-dependence permeability and viscosities.

## 7 Global non linear stability via symmetries and skew-symmetries hidden in (3.6)

Setting  $\Psi_1 = R_1\theta - P_1R\Phi_1$ ,  $\Psi_2 = R_2\theta + P_2R\Phi_2$ ; it follows that (3.6) is equivalent to  $(\alpha = 1, 2)$

$$\begin{cases} \nabla\Pi = -f(z)\mathbf{u} + \frac{1}{R} \left( R^*\theta + \frac{R_1}{P_1}\Psi_1 - \frac{R_2}{P_2}\Psi_2 \right) \mathbf{e}_3 + 2D_a \sum_{i,j}^{1-3} \frac{\partial}{\partial x_j} (gD_{ij}) \mathbf{e}_i, \\ \nabla \cdot \mathbf{u} = 0, \quad \frac{d\theta}{dt} = Rw + \Delta\theta, \quad P_\alpha \frac{d\Psi_\alpha}{dt} = \Delta\Psi_\alpha + R_\alpha(P_\alpha - 1)\Delta\theta, \end{cases} \quad (7.1)$$

under the boundary conditions

$$w = \theta = \Psi_\alpha = 0, \quad \text{on } z = 0, 1, \quad (7.2)$$

with

$$R^* = R^2 - \frac{R_1^2}{P_1} + \frac{R_2^2}{P_2}. \quad (7.3)$$

System (7.1)<sub>1</sub>-(7.2)<sub>2</sub> together with (7.3) is reducible to (3.6) via the substitution

$$\begin{pmatrix} \frac{R^*}{R} & \frac{R_1}{P_1 R} & -\frac{R_2}{P_2 R} & \Psi_1 & \Psi_2 \\ R & -R_1 & -R_2 & \Phi_1 & \Phi_2 \end{pmatrix} \quad (7.4)$$

and therefore it follows that

$$\frac{\partial}{\partial t} \begin{pmatrix} \theta \\ \Psi_1 \\ \Psi_2 \end{pmatrix} = \sum_{n=1}^{\infty} \mathcal{L}_n^* \begin{pmatrix} \theta_n \\ \Psi_{1n} \\ \Psi_{2n} \end{pmatrix} - \mathbf{u} \cdot \nabla \begin{pmatrix} \theta \\ \Psi_1 \\ \Psi_2 \end{pmatrix}, \quad (7.5)$$

with

$$\mathcal{L}_n^* = \begin{pmatrix} R^* \tilde{\mathcal{A}}_n - \xi_n & \frac{R_1}{P_1} \tilde{\mathcal{A}}_n & -\frac{R_2}{P_2} \tilde{\mathcal{A}}_n \\ -\frac{R_1}{P_1} (P_1 - 1) \xi_n & -\frac{\xi_n}{P_1} & 0 \\ -\frac{R_2}{P_2} (P_2 - 1) \xi_n & 0 & -\frac{\xi_n}{P_2} \end{pmatrix}. \quad (7.6)$$

**Theorem 7.1** *The global non linear stability of the conduction*

solution is guaranteed by

$$R^2 < R_1^2 - R_2^2 + H^* = R_{C_3}, \quad \text{for } P_1 \leq 1, \ P_2 \geq 1, \quad (7.7)$$

$$R^2 < \frac{R_1^2}{P_1} - \frac{R_2^2}{P_2} + H^*, \quad \text{for } P_1 \geq 1, \ P_2 \leq 1, \quad (7.8)$$

$$R^2 < R_1^2 - \frac{R_2^2}{P_2} + H^*, \quad \text{for } P_1 \leq 1, \ P_2 \leq 1, \quad (7.9)$$

$$R^2 < \frac{R_1^2}{P_1} - R_2^2 + H^*, \quad \text{for } P_1 \geq 1, \ P_2 \geq 1, \quad (7.10)$$

where  $H^*$  is given by (5.10).

**Proof.** Since the operator  $\mathcal{L}_n^*$  can be obtained by the operator  $\mathcal{L}_n$  appearing in (6.6) of [23] by putting  $\tilde{\mathcal{A}}_n$  at the place of  $\eta_n$ , following step by step the proof given in [23], the theorem immediately follows.

For the sake of completeness, we give here a sketch of the proof of (7.7). By virtue of the absence of subcritical instabilities, (7.5)

reduces to

$$\frac{\partial}{\partial t} \begin{pmatrix} \theta_n \\ \psi_{1n} \\ \psi_{2n} \end{pmatrix} = \mathcal{L}_n^* \begin{pmatrix} \theta_n \\ \psi_{1n} \\ \psi_{2n} \end{pmatrix}. \quad (7.11)$$

Setting

$$\psi_{\alpha n} = \sqrt{\frac{|P_\alpha - 1| \xi_n}{\tilde{\mathcal{A}}_n}} \varphi_{\alpha n}, \quad (\alpha = 1, 2) \quad (7.12)$$

(7.11) for  $(P_1 \leq 1, P_2 \geq 1)$  becomes

$$\frac{\partial}{\partial t} \begin{pmatrix} \theta_n \\ \psi_{1n} \\ \psi_{2n} \end{pmatrix} = \tilde{\mathcal{L}}_n \begin{pmatrix} \theta_n \\ \psi_{1n} \\ \psi_{2n} \end{pmatrix} \quad (7.13)$$



with  $\tilde{\mathcal{L}}_n$  *symmetric* operator given by

$$\tilde{\mathcal{L}}_n = \begin{pmatrix} R^* \tilde{\mathcal{A}}_n - \xi_n & \frac{R_1}{P_1} \sqrt{(1 - P_1) \xi_n \tilde{\mathcal{A}}_n} & -\frac{R_2}{P_2} \sqrt{(P_2 - 1) \xi_n \tilde{\mathcal{A}}_n} \\ \frac{R_1}{P_1} \sqrt{(1 - P_1) \xi_n \tilde{\mathcal{A}}_n} & -\frac{\xi_n}{P_1} & 0 \\ -\frac{R_2}{P_2} \sqrt{(P_2 - 1) \xi_n \tilde{\mathcal{A}}_n} & 0 & -\frac{\xi_n}{P_2} \end{pmatrix}. \quad (7.14)$$

Since,  $\forall n \in \mathbb{N}$ , the eigenvalues of  $\tilde{\mathcal{L}}_n$  are real numbers, the marginal state is a stationary state and the critical Rayleigh number is given by the invariant  $R_{C_3}$  and, in view of (5.12) for  $\alpha = 3$ , (7.7) immediately follows.

## 8 Applications

By virtue of (5.11)-(5.13) and (7.7)-(7.10), the influence of depth-dependence on the onset of convection, is measured via  $H^*$ : the onset of convection is delayed by the increasing of  $H^*$ . We now, for

the sake of simplicity, confine ourselves to the case

$$g = 1 + bz, \quad g \geq 0 \text{ in } (0, 1), \quad (8.1)$$

with  $b$  real constant. In view of (8.1) and

$$\int_0^1 g' \sin 2n\pi z \, dz = \frac{b}{n\pi} \int_0^1 \frac{d}{dz} (\sin^2 n\pi z) \, dz = 0, \quad (8.2)$$

it follows that

$$\begin{cases} A_n^* = \int_0^1 (n^2 \pi^2 \cos^2 n\pi z + a^2 \sin^2 n\pi z) f \, dz, \\ B_n^* = \xi_n^2 \int_0^1 g \sin^2 n\pi z \, dz, \\ H_n = \frac{2\xi_n}{\tilde{A}_n} = \frac{2}{a^2} \xi_n (A_n^* + B_n^*) > 2\xi_n \int_0^1 \left( f + \frac{\xi_n^2}{a^2} g \right) \sin^2 n\pi z \, dz. \end{cases} \quad (8.3)$$

Therefore

$$H^* = \min_{(a^2, n) \in \mathbb{R}^+ \times \mathbb{N}} H_n = \min_{a^2 \in \mathbb{R}^+} H_1. \quad (8.4)$$

In view of

$$\lim_{a^2 \rightarrow 0} H_1 = \lim_{a^2 \rightarrow \infty} H_1 = \infty, \quad (8.5)$$

it follows that there exists a positive bounded critical value  $a_c^2 > 0$  such that

$$H^* = (H_1)_{a^2=a_c^2}. \quad (8.6)$$

Setting

$$h_1 = \int_0^1 f \sin^2 \pi z dz, \quad h_2 = \int_0^1 f \cos^2 \pi z dz, \quad h_3 = \frac{B_1^*}{a^2 + \pi^2}, \quad (8.7)$$

it easily follows that

$$\frac{\partial H_1}{\partial a^2} = 0 \Leftrightarrow \pi^2(a^2 h_1 + \pi^2 h_2) = a^2(a^2 + \pi^2)(h_1 + h_3) \quad (8.8)$$

and hence

$$a_c^2 = \frac{\pi^2}{2(h_1 + h_3)} \left\{ [h_3^2 + 4h_2(h_1 + h_3)]^{\frac{1}{2}} - h_3 \right\}. \quad (8.9)$$

Denoting by  $H_u^*$  and  $H_l^*$  an upper and a lower bound of  $H^*$ , in view of

$$\begin{cases} \int_0^1 \sin^2 \pi z \, dz = \int_0^1 \cos^2 \pi z \, dz = \frac{1}{2}, \\ \int_0^1 g \sin^2 \pi z \, dz = \frac{1}{2} + b \int_0^1 z \sin^2 \pi z \, dz, \end{cases} \quad (8.10)$$

one obtains

$$\begin{cases} H_l^* = \frac{2}{a_c^2}(a_c^2 + \pi^2) \left[ \frac{(\pi^2 + a_c^2)\bar{f}}{2} + (a_c^2 + \pi^2)^2 \frac{\bar{g}}{2} \right] > \\ > (\pi^2 + a_c^2)[\bar{f} + (a_c^2 + \pi^2)\bar{g}] > \pi^2(\bar{f} + \pi^2\bar{g}), \\ H_u^* = \frac{(a_c^2 + \pi^2)^2}{a_c^2}[\bar{\bar{f}} + (a_c^2 + \pi^2)\bar{\bar{g}}], \end{cases} \quad (8.11)$$

with  $\bar{f} = \text{ess inf}_{(0,1)} f$ ,  $\bar{\bar{f}} = \text{ess sup}_{(0,1)} f$ ,  $\bar{g} = \text{ess inf } g > 0$ ,  $\bar{\bar{g}} = \text{ess sup}_{(0,1)} g$ .

### *7.1 Applications to the earth's mantle*

The increase in viscosity  $\mu_1$  in earth's mantle proposed by Torrance

and Turgotte (see [1], p.118) is given by

$$\mu_1 = \bar{\mu}_1 f_1(z), \quad f_1(z) = e^{c(1/2-z)}, \quad c = \text{const.} > 0. \quad (8.12)$$

For a linear variation in the permeability given by

$$K = \bar{K} f_3(z), \quad f_3(z) = 1 + \gamma z, \quad \gamma = \text{const.}, \quad (8.13)$$

one obtains

$$f = \frac{e^{c(1/2-z)}}{1 + \gamma z}. \quad (8.14)$$

The corresponding values of  $H^*$ , for various values of  $c, \gamma, b$  are furnished by the following table.

### *7.2 Stratification of large pores in artificial porous materials*

The stratification of pores in artificial porous materials has a very relevant interest. In fact, for instance, in the construction of porous materials for insulating purposes, the aim is to delay or prohibit heat transfer and hence high thermal critical Rayleigh numbers are requested. Viceversa in cooling pipes used in computers or in other modern devices, the porosity has to be stratified in such a way to

Table 1: Values of  $H^*$  for the earth's mantle

f	g	$h_1$	$h_2$	$h_3$	$a_c^*$	$H^*$
$\frac{e^{(0.5-z)}}{1+0.5z}$	$1+0.5z$	0.41	0.46	0.625	4.23	100.61
$\frac{e^{(0.5-z)}}{1-0.5z}$	$1+0.2z$	0.67	0.69	0.55	5.53	105.94
$\frac{e^{0.5(0.5-z)}}{1-0.2z}$	$1+0.5z$	0.68	0.70	0.625	5.25	114.64
$\frac{e^{0.5(0.5-z)}}{1+1.5z}$	$1+z$	0.30	0.34	0.75	3.12	116.95
$\frac{e^{0.5(0.5-z)}}{1-0.2z}$	$1+1.5z$	0.56	0.56	0.875	3.86	140.16
$\frac{e^{2.5(0.5-z)}}{1-0.8z}$	$1+1.5z$	0.88	1.03	0.875	5.50	159.07
$\frac{e^{(0.5-z)}}{1+2.5z}$	$1+2z$	0.25	0.33	1	2.49	161.28
$\frac{e^{0.5(0.5-z)}}{1+2z}$	$1+2.5z$	0.26	0.32	1.125	2.18	190.83

produce low thermal critical Rayleigh numbers. We now confine ourselves to the stratification of large pores in porous artificial materials assuming  $f = 1$  and  $\bar{g}$  given by (8.1).

Values of  $g$  guaranteeing increasing values of  $H^*$  and hence high

thermal critical Rayleigh numbers are furnished in table 2.

Table 2: Stratification of large pores for prohibiting heat transfer

f	g	$h_1 = h_2$	$h_3$	$a_c^*$	$H^*$
1	$1+0.1z$	$1/2$	0.525	4.81444	91.812
1	$1+0.3z$	$1/2$	0.575	4.59051	97.9331
1	$1+0.5z$	$1/2$	0.625	4.38649	104.248
1	$1+z$	$1/2$	$3/4$	3.94784	120.903
1	$1+1.5z$	$1/2$	0.875	3.58895	138.791
1	$1+2z$	$1/2$	1	3.28987	157.914
1	$1+3.5z$	$1/2$	1.375	2.63189	222.683
1	$1+5z$	$1/2$	1.75	2.19325	298.556
1	$1+7z$	$1/2$	2.25	1.79447	416.991
1	$1+10z$	$1/2$	3	1.41	631.655
1	$1+20z$	$1/2$	5.5	0.822	1667.96

Values of  $g$  guaranteeing decreasing values of  $H^*$  and hence low thermal critical Rayleigh numbers are furnished in table 3.

Table 3: Stratification of large pores for rapid heat transfer

f	g	$h_1 = h_2$	$h_3$	$a_c^*$	$H^*$
1	1-0.1z	1/2	0.475	5.06134	85.8902
1	1-1/5z	1/2	0.45	5.19453	83.0034
1	1-1/3z	1/2	0.416667	5.383422	79.231
1	1-1/2z	1/2	0.375	5.63977	74.6389
1	1-2/3z	1/2	0.33333	5.92176	70.1839
1	1-3/4z	1/2	0.3125	6.0736	68.0077
1	1-7/8z	1/2	0.28125	6.31655	64.8078
1	1-7/8z	1/2	0.28125	6.31655	64.8078