A control problem for the steady self-propelled motion of a rigid body in a Navier-Stokes fluid

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Workshop on PDE's and Biomedical Applications, December 4-6, 2014, Lisbon

Outline

Mathematical formulation of self-propelled motion of a rigid body in a fluid

- Equations of self-propelled motion in a Navier-Stokes liquid (direct and control problems)
- The Stokes approximation of self-propelled motions (direct and control problems)

A generalized Oseen problem

- Existence and uniqueness of solution and corresponding estimates

Motion control of a self-propelled rigid body

- Formulation of the control problem
- A linearized version of the control problem
- The non-linear control problem

The fluid structure interaction problem

We shall say that a rigid body $\mathcal{S}$ undergoes a self-propelled motion in a fluid $\mathcal{F}$ if
(ii) the total external force acting on $\mathcal{F}$ is identically zero,
(iii) the total net force and torque, external to $\{\mathcal{F}, \mathcal{S}\}$, acting on $\mathcal{S}$ are identically zero.

In the absence of external actions, the forward force (thrust) that makes the body move is generated by the body, and the motion is due to the interaction of the body's external surface and the fluid in which it is immersed.

In practice, such a velocity can be produced by propellers (submarines), deformations (fishes), cilia (micro-organisms), etc.

The fluid structure interaction problem

Let

$$V(x) := \xi + \omega \times x$$

$$\sigma(v,p) := 2D(v) - pI_3 = \nabla v - (\nabla v)^T - pI_3$$

The direct/classical steady problem is: given v_*, find (V, v, p) satisfying

$$-\text{div}\,\sigma(v,p) + (v - \xi - \omega \times x)\cdot \nabla v + \omega \times v = 0 \quad \text{in } \Omega$$

$$\text{div}\, v = 0 \quad \text{in } \Omega$$

$$v = \xi + \omega \times x + v_* \quad \text{on } \partial\Omega$$

$$\lim_{|x|\to\infty} v = 0$$

$$m\xi \times \omega + \int_{\partial\Omega} \left[-\sigma(v,p)n + (v_* \cdot n)(v_* + V)\right] \, d\gamma = 0$$

$$(I\omega)\times\omega + \int_{\partial\Omega} x \times \left[-\sigma(v,p)n + (v_* \cdot n)(v_* + V)\right] \, d\gamma = 0$$

The inverse/control fluid structure interaction problem

How should the boundary velocity v_* be prescribed in order to produce a desired velocity $V(x) = \xi + \omega \times x$ of $\mathcal{R}$?

In this case, ξ and ω are known, and we have to find (v_*, v, p) satisfying

$$-\text{div } \sigma(v,p) + (v - \xi - \omega \times x) \cdot \nabla v + \omega \times v = 0 \quad \text{in } \Omega$$

$$\text{div } v = 0 \quad \text{in } \Omega$$

$$v = \xi + \omega \times x + v_* \quad \text{on } \partial\Omega$$

$$\lim_{|x| \to \infty} v = 0$$

$$m\xi \times \omega + \int_{\partial\Omega} \left[-\sigma(v,p)n + (v_* \cdot n)(v_* + V)\right] \, d\gamma = 0$$

$$(I\omega) \times \omega + \int_{\partial\Omega} x \times \left[-\sigma(v,p)n + (v_* \cdot n)(v_* + V)\right] \, d\gamma = 0$$

The Stokes approximation

Neglecting all non-linear terms in the previous system yields

$$\left.\begin{aligned} \text{div } \sigma(v,p) &= 0 \\ \text{div } v &= 0 \end{aligned}\right\} \text{ in } \Omega$$

$$v = v_* + V \text{ on } \partial\Omega$$

$$\lim_{|x|\to\infty} v(x) = 0$$

$$\int_{\partial\Omega} \sigma(v,p)\cdot n = 0$$

$$\int_{\partial\Omega} x \times \sigma(v,p)\cdot n = 0$$

Auxiliary Stokes problems associated with elementary rigid motions - translations

Let (e_1, e_2, e_3) be the canonical basis of $\mathbb{R}^3$.
For each $i \in \{1, 2, 3\}$, $(v_0^{(i)}, q_0^{(i)})$ is the solution of

$$-\text{div } \sigma(v_0^{(i)}, q_0^{(i)}) = 0 \quad \text{in } \Omega$$

$$\text{div } v_0^{(i)} = 0 \quad \text{in } \Omega$$

$$v_0^{(i)} = e_i \quad \text{on } \partial\Omega$$

$$\lim_{|x| \to \infty} v_0^{(i)}(x) = 0$$

The corresponding thrust functions are

$$g_0^{(i)} := \sigma(v_0^{(i)}, q_0^{(i)}) n|_{\partial\Omega}, \quad i = 1, 2, 3.$$

Auxiliary Stokes problems associated with elementary rigid motions - rotations

For $i \in \{1, 2, 3\}$, $(V_0^{(i)}, Q_0^{(i)})$ is the solution of

$$-\text{div } \sigma(V_0^{(i)}, Q_0^{(i)}) = 0 \quad \text{in } \Omega$$

$$\text{div } V_0^{(i)} = 0 \quad \text{in } \Omega$$

$$V_0^{(i)} = e_i \times x \quad \text{on } \partial\Omega$$

$$\lim_{|x| \to \infty} V_0^{(i)}(x) = 0$$

The corresponding thrust functions are

$$G_0^{(i)} := \sigma(V_0^{(i)}, Q_0^{(i)}) \cdot n_{|\partial\Omega}, \quad i = 1, 2, 3.$$

Stokes approximation - Direct problem

The motion of the body can be completely decoupled from that of the liquid by reducing the calculation of ξ and ω to a linear algebraic system

$$R\begin{bmatrix}\xi\\ \omega\end{bmatrix}=\begin{bmatrix}a\\ b\end{bmatrix}$$

where R is the *resistance matrix*

$$R_{ij}=\int_{\partial\Omega} e_j\cdot g_0^{(i)}, \qquad R_{i+3,j}=\int_{\partial\Omega} e_j\cdot G_0^{(j)}, \quad i,j=1,2,3,$$

$$R_{i,j+3}=\int_{\partial\Omega} e_j\times x\cdot g_0^{(i)}, \qquad R_{i+3,j+3}=\int_{\partial\Omega} e_j\times x\cdot G_0^{(i)}, \quad i,j=1,2,3,$$

and

$$a_i=-\int_{\partial\Omega} v_*\cdot g_0^{(i)}, \quad i=1,2,3, \qquad b_i=-\int_{\partial\Omega} v_*\cdot G_0^{(i)}, \quad i=1,2,3.$$

The role of the thrust functions in the Stokes approximation

The resolution of the Stokes problem by reducing it to a linear system allows a complete knowledge of the relation between the thrust v_* and the resulting velocity V. In particular

$$V \neq 0 \Longleftrightarrow P_{\mathcal{C}}(v_*) \neq 0.$$

where

$$\mathcal{C} = \text{span }\{g_0^{(1)}, g_0^{(2)}, g_0^{(3)}, G_0^{(1)}, G_0^{(2)}, G_0^{(3)}\}$$

The self-propelling conditions are equivalent to

$$\int_{\partial\Omega} (v_* + V) \cdot g_0^{(i)} = 0, \quad i = 1, 2, 3$$

$$\int_{\partial\Omega} (v_* + V) \cdot G_0^{(i)} = 0, \quad i = 1, 2, 3$$

Stokes approximation - Control spaces and adjoint problems

$$\mathcal{C} = \text{span}\ \{g_0^{(i)}, G_0^{(i)}\ ;\ i = 1, 2, 3\}$$

$$\mathcal{C}_\tau = \text{span}\ \{(g_0^{(i)} \times n) \times n, (G_0^{(i)} \times n) \times n\ ;\ i = 1, 2, 3\}$$

The linear control problem for v_* can be solved in $\mathcal{C}$ and $\mathcal{C}_\tau$ with the aid of the following systems

$$\begin{aligned} -\text{div}\ \sigma(u_0^{(i)}, q_0^{(i)}) &= 0 \\ \text{div}\ u_0^{(i)} &= 0 \\ u_0^{(i)} = g_0^{(i)} &\quad \text{on } \partial\Omega \\ \lim_{|x|\to\infty} u_0^{(i)} &= 0, \end{aligned} \qquad \begin{aligned} -\text{div}\ \sigma(U_0^{(i)}, Q_0^{(i)}) &= 0 \\ \text{div}\ U_0^{(i)} &= 0 \\ U_0^{(i)} = G_0^{(i)} &\quad \text{on } \partial\Omega \\ \lim_{|x|\to\infty} U_0^{(i)} &= 0 \end{aligned}$$

Stokes approximation - Control problem

Formulation as a linear algebraic system

$$A_{(0,0)} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \delta \\ \zeta \end{pmatrix},$$

can be done using the matrix $A := A_{(0,0)} \in \mathbb{R}^{6\times 6}$ is defined by

$$\begin{aligned}
A_{i,j} &= \int_{\partial\Omega} g_0^{(i)} \cdot g_0^{(j)} \, d\gamma \quad (i,j \leq 3), \\
A_{i,j} &= \int_{\partial\Omega} g_0^{(i)} \cdot G_0^{(j-3)} \, d\gamma \quad (i \leq 3, j \geq 4), \\
A_{i,j} &= \int_{\partial\Omega} G_0^{(i-3)} \cdot g_0^{(j)} \, d\gamma \quad (i \geq 4, j \leq 3), \\
A_{i,j} &= \int_{\partial\Omega} G_0^{(i-3)} \cdot G_0^{(j-3)} \, d\gamma \quad (i,j \geq 4).
\end{aligned}$$

Lemma A is symmetric and positive definite.

Classical Oseen problem

$$-\text{div}\ \sigma(v,p) = \zeta\cdot\nabla v + f \quad \text{in } \Omega$$

$$\text{div}\ v = 0 \quad \text{in } \Omega$$

$$v = v_* \quad \text{on } \partial\Omega$$

$$\lim_{|x|\to\infty} v(x) = 0$$

There is an infinite paraboloidal region within which v decays like $|x|^{-1}$ and outside of which v decays even faster. This non-uniform decay is representative of the wake behind the body and is described by

$$w(x) = (1+|x|)\,(1+2(|x||\zeta|+\zeta\cdot x))\,.$$

Generalized Oseen problem

$$-\text{div}\,\sigma(v,p) = (\zeta + \theta \times x)\cdot\nabla v - \theta \times v + f \quad \text{in } \Omega$$

$$\text{div } v = 0 \quad \text{in } \Omega$$

$$v = v_* \quad \text{on } \partial\Omega$$

$$\lim_{|x|\to\infty} v(x) = 0$$

In the case $\zeta \neq 0$, it is expected that v decays faster outside a paraboloidal region behind the body, representative of the wake. This is described by

$$w(x) = \left(1 + \left|x - \frac{\theta \times \zeta}{|\theta|^2}\right|\right)\left[1 + 2\frac{|\theta\cdot\zeta|}{|\theta|}\left(\left|x - \frac{\theta \times \zeta}{|\theta|^2}\right| + x\cdot\frac{\theta}{|\theta|}\right)\right].$$

There is no wake if ζ and θ are orthogonal, and if $\theta \neq 0$ and $\theta\cdot\zeta \neq 0$ there is a formation of a wake (along the direction of θ) whose "width" will depend on the angle between θ and ζ.

Well-posedness of the generalized Oseen problem

Theorem Assume that $\partial\Omega$ is of class C^2, $f = \nabla \cdot F \in L^2(\Omega)$, with

$$\lceil F \rceil_{2,w,\Omega} := \sup_{x\in\Omega} \left[w(x)^2 |F(x)| \right] < \infty$$

and $v_* \in W^{3/2,2}(\partial\Omega)$. Then, there exists a unique solution (v, p) to the generalized Oseen problem with

$$\nabla v \in W^{1,2}(\Omega), \quad p \in W^{1,2}(\Omega),$$

$$\lceil v \rceil_{1,w,\Omega} := \sup_{x\in\Omega} [w(x)|v(x)|] < \infty$$

and

$$|v|_{2,2,\Omega} + |v|_{1,2,\Omega} + \lceil v \rceil_{1,\omega,\Omega} + \|p\|_{1,2,\Omega} \leq C(\|\nabla \cdot F\|_{2,\Omega} + \lceil F \rceil_{2,\omega,\Omega} + \|v_*\|_{3/2,2,\partial\Omega})$$

In the above estimate, if $|\zeta|, |\theta| \in [0, B]$, one can choose C independent of ζ and θ.

Extension of the boundary velocity $V(x) = \xi + \omega \times x$

The extension $E \in C_0^\infty(\overline{\Omega})$ of V can be defined by

$$E(x) = \frac{1}{2}\nabla \times \left[\eta(x)(\xi \times x - |x|^2\omega)\right],$$

where $\eta \in C_0^\infty(\mathbb{R}^3)$ is such that $0 \leq \eta \leq 1$ and $\eta = 1$ in a neighborhood of the body $\mathcal{S}$.

Setting $u := v - E$, the control problem consists in finding v_* in an apropriate space such that

$$\begin{gathered}
-\mathrm{div}\ \sigma(u,p) - (\xi + \omega \times x) \cdot \nabla u + \omega \times u = f(u) \quad \text{in } \Omega \\
\mathrm{div}\ u = 0 \quad \text{in } \Omega \\
u = v_* \quad \text{on } \partial\Omega \\
\lim_{|x|\to\infty} u = 0 \\
-\int_{\partial\Omega} [\sigma(u,p)n + (\xi + \omega \times x) \cdot nu] = \xi_f(v_*) \\
-\int_{\partial\Omega} x \times [\sigma(u,p)n + (\xi + \omega \times x) \cdot nu]\, d\gamma = \omega_f(v_*),
\end{gathered}$$

Extension of the boundary velocity $V(x) = \xi + \omega \times x$

$$f(u) := -(u\cdot\nabla u + u\cdot\nabla E + E\cdot\nabla u) - [(E-V)\cdot\nabla E + \omega\times E] + \Delta E,$$

$$\xi_f(v_*) := -\int_{\partial\Omega}[(v_*+V)v_*\cdot n + (V\cdot n)v_*]d\gamma + 2\int_{\partial\Omega} D(E)nd\gamma - m\xi\times\omega,$$

$$\omega_f(v_*) := -\int_{\partial\Omega} x\times[(v_*+V)v_*\cdot n + (V\cdot n)v_*]d\gamma + 2\int_{\partial\Omega} x\times D(E)nd\gamma - (I\omega)\times\omega.$$

Note that

$$f(u) = \nabla\cdot F(u),$$

with

$$F(u) := 2D(E) + E\otimes(\xi+\omega\times x - E) - u\otimes u - u\otimes E - E\otimes u - (\omega\times x)\otimes E.$$

Adjoint systems

For each $i \in \{1,2,3\}$, let $(v^{(i)}, q^{(i)})$ be the solution of the generalized Oseen problem

$$-\text{div}\ \sigma(v^{(i)}, q^{(i)}) + (\xi + \omega \times x) \cdot \nabla v^{(i)} - \omega \times v^{(i)} = 0 \quad \text{in } \Omega$$

$$\text{div}\ v^{(i)} = 0 \quad \text{in } \Omega$$

$$v^{(i)} = e_i \quad \text{on } \partial\Omega$$

$$\lim_{|x|\to\infty} v^{(i)}(x) = 0$$

where (e_1, e_2, e_3) is the canonical basis of $\mathbb{R}^3$. We also consider the solutions $(V^{(i)}, Q^{(i)})$ of

$$-\text{div}\ \sigma(V^{(i)}, Q^{(i)}) + (\xi + \omega \times x) \cdot \nabla V^{(i)} - \omega \times V^{(i)} = 0 \quad \text{in } \Omega$$

$$\text{div}\ V^{(i)} = 0 \quad \text{in } \Omega$$

$$V^{(i)} = e_i \times x \quad \text{on } \partial\Omega$$

$$\lim_{|x|\to\infty} V^{(i)}(x) = 0$$

Adjoint systems and the control spaces

Lemma Assume Ω is of class C^2. There exist unique solutions $(v^{(i)}, p^{(i)})$ and $(V^{(i)}, P^{(i)})$ of the adjoint systems. Moreover, there exists a constant C independent of ξ and ω such that for $|\xi|, |\omega| \in [0, B]$, and for $i \in \{1, 2, 3\}$

$$|v^{(i)}|_{2,2,\Omega} + |v^{(i)}|_{1,2,\Omega} + \lceil v^{(i)} \rceil_{1,w,\Omega} + \|q^{(i)}\|_{1,2,\Omega} \leq C,$$

$$|V^{(i)}|_{2,2,\Omega} + |V^{(i)}|_{1,2,\Omega} + \lceil V^{(i)} \rceil_{1,w,\Omega} + \|Q^{(i)}\|_{1,2,\Omega} \leq C.$$

We define

$$g^{(i)} := \sigma(v^{(i)}, q^{(i)})n \quad \text{on } \partial\Omega,$$

$$G^{(i)} := \sigma(V^{(i)}, Q^{(i)})n \quad \text{on } \partial\Omega.$$

and the control spaces

$$\mathcal{C} := \text{span } \{g^{(i)}, G^{(i)}; i = 1, 2, 3\}$$

$$\mathcal{C}_\tau := \{(g^{(i)} \times n) \times n, (G^{(i)} \times n) \times n \; ; \; i = 1, 2, 3\}$$

The linear problem

Given $V(x) = \xi + \omega \times x$, find $v_* \in \mathcal{C}$(or $\mathcal{C}_\tau$) and the corresponding (u, q) satisfying

$$-\text{div}\ \sigma(u,q) - (\xi + \omega \times x) \cdot \nabla u + \omega \times u = f \quad \text{in } \Omega$$

$$\text{div}\ u = 0 \quad \text{in } \Omega$$

$$u = \sum_{i=1}^{3} \alpha_i g^{(i)} + \beta_i G^{(i)} \quad \text{on } \partial\Omega$$

$$\lim_{|x| \to \infty} u = 0$$

$$-\int_{\partial\Omega} [\sigma(u,q)n + (\xi + \omega \times x) \cdot nu]\ d\gamma = \xi_f$$

$$-\int_{\partial\Omega} x \times [\sigma(u,q)n + (\xi + \omega \times x) \cdot nu]\ d\gamma = \omega_f.$$

where f, ξ_f and ω_f are also given.

Auxiliary linear systems for the control problem

The linear control problem will be solved with the aid of the following systems

$$-\text{div}\ \sigma(u^{(i)}, q^{(i)}) + \omega \times u^{(i)} - (\xi + \omega \times x) \cdot \nabla u^{(i)} = 0$$

$$\text{div}\ u^{(i)} = 0$$

$$u^{(i)} = g^{(i)} \quad \text{on } \partial\Omega$$

$$\lim_{|x| \to \infty} u^{(i)} = 0,$$

$$-\text{div}\ \sigma(U^{(i)}, Q^{(i)}) + \omega \times U^{(i)} - (\xi + \omega \times x) \cdot \nabla U^{(i)} = 0$$

$$\text{div}\ U^{(i)} = 0$$

$$U^{(i)} = G^{(i)} \quad \text{on } \partial\Omega$$

$$\lim_{|x| \to \infty} U^{(i)} = 0$$

Auxiliary linear systems for the control problem

and

$$-\text{div}\ \sigma(u_f, q_f) + \omega \times u_f - (\xi + \omega \times x) \cdot \nabla u_f = f$$

$$\text{div}\ u_f = 0$$

$$u_f = 0 \quad \text{on } \partial\Omega$$

$$\lim_{|x| \to \infty} u_f = 0$$

Well-posedness of the auxiliary linear problems

Lemma Assume Ω is of class C^4. For each $i \in \{1,2,3\}$, there exist unique solutions $(u^{(i)}, p^{(i)})$ and $(U^{(i)}, P^{(i)})$ of the auxiliary problems and there exists a constant C independent of ξ and ω such that

$$|u^{(i)}|_{2,2,\Omega} + |u^{(i)}|_{1,2,\Omega} + \lceil u^{(i)} \rceil_{1,w,\Omega} + \|p^{(i)}\|_{1,2,\Omega} \leq C,$$

$$|U^{(i)}|_{2,2,\Omega} + |U^{(i)}|_{1,2,\Omega} + \lceil U^{(i)} \rceil_{1,w,\Omega} + \|P^{(i)}\|_{1,2,\Omega} \leq C.$$

provided $|\xi|, |\omega| \in [0, B]$. There exist a unique solution (u_f, p_f) and there exists a constant C independent of ξ and ω such that

$$|u_f|_{2,2,\Omega} + |u_f|_{1,2,\Omega} + \lceil u_f \rceil_{1,w,\Omega} + \|p_f\|_{1,2,\Omega} \leq C \left(\|f\|_{2,\Omega} + \lceil F \rceil_{2,w,\Omega} \right).$$

whenever $|\xi|, |\omega| \in [0, B]$.

Formulation as a linear algebraic system

$$A \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \delta \\ \zeta \end{pmatrix},$$

where

$$\delta_j := \xi_f \cdot e_j + \int_{\partial\Omega} \sigma(u_f, p_f) n \, d\gamma \cdot e_j \quad (j = 1, 2, 3),$$

$$\eta_j := \omega_f \cdot e_j + \int_{\partial\Omega} x \times \sigma(u_f, p_f) n \, d\gamma \cdot e_j \quad (j = 1, 2, 3),$$

and $A \in \mathbb{R}^{6\times 6}$ is defined by

$$A_{i,j} = \int_{\partial\Omega} g^{(i)} \cdot g^{(j)} \, d\gamma \quad (i, j \leq 3),$$

$$A_{i,j} = \int_{\partial\Omega} g^{(i)} \cdot G^{(j-3)} \, d\gamma \quad (i \leq 3, j \geq 4),$$

$$A_{i,j} = \int_{\partial\Omega} G^{(i-3)} \cdot g^{(j)} \, d\gamma \quad (i \geq 4, j \leq 3),$$

$$A_{i,j} = \int_{\partial\Omega} G^{(i-3)} \cdot G^{(j-3)} \, d\gamma \quad (i, j \geq 4).$$

Formulation as a linear algebraic system

Lemma The matrix A is symmetric. There exist two positive constant $c_1 = c_1(\Omega)$ and $c_2 = c_2(\Omega)$ such that if $|\xi| + |\omega| \leq c_1$ then A is definite positive with

$$\|A^{-1}\|_{\mathcal{L}(\mathbb{R}^6)} \leq c_2.$$

Lemma The matrix A_τ is symmetric. There exist three positive constant $c_1 = c_1(\Omega)$, $c_2 = c_2(\Omega)$ and $c_3 = c_3(\Omega)$ such that if $|\xi| \leq c_1|\omega| \leq c_2$, then A_τ is definite positive with

$$\|A_\tau^{-1}\|_{\mathcal{L}(\mathbb{R}^6)} \leq c_3.$$

Proof: Continuity of $A_{(\xi,\omega)}$ in $(0,0)$ and invertibility of $A_{(0,0)}$. Uses the convergence of the generalized Oseen system to the Stokes system when $(\xi,\omega) \to (0,0)$.

The main result

Theorem Assume $\partial\Omega$ is of class C^4 and $\xi, \omega \in \mathbb{R}^3$ are *given*. There exists $C_1, C_2 > 0$ such that if $|\xi| \leq C_1|\omega| \leq C_2$ then the control problem admits a solution (v, p, v_*). More specifically, under the above assumptions on the data, there exists a 6-dimensional control space $\mathcal{C} \subset W^{3/2,2}(\partial\Omega)$ (or $\mathcal{C}_\tau$) such that a unique solution (v, p, v_*) can be found with

$$v\varpi \in L^\infty(\Omega),$$

$$\nabla v \in W^{1,2}(\Omega),\ p \in W^{1,2}(\Omega),\ v_* \in \mathcal{C},$$

where

$$\varpi(x) := \begin{cases} (1+|x|)\left[1 + 2\frac{|\omega\cdot\xi|}{|\omega|}\left(|x| + x\cdot\frac{\omega}{|\omega|}\right)\right], & \omega \neq 0, \\ (1+|x|)\left(1+2(|x||\xi| + \xi\cdot x)\right), & \omega = 0. \end{cases}$$

Proof and References

Consider

$$\mathcal{X} := \{(f, \xi_f, \omega_f) \in L^2(\Omega) \times \mathbb{R}^6 \; ; \; f = \nabla \cdot F, \quad \lceil F \rceil_{2,w} < \infty\}$$

endowed with the norm

$$\|(f, \xi_f, \omega_f)\|_{\mathcal{X}} := \|f\|_{L^2(\Omega)} + \lceil F \rceil_{2,w} + \|(\xi_f, \omega_f)\|_{\mathbb{R}^6}$$

and the mapping $\mathcal{Z}(f, \xi_f, \omega_f) = (\widehat{f}, \widehat{\xi}_f, \widehat{\omega}_f)$ where $f = \widehat{f}$, $\xi_f = \widehat{\xi}_f$, $\omega_f = \widehat{\omega}_f$.

The result is obtained by Banach fixed point theorem.

Some references

Galdi (1997,1999,2011), Galdi and S. (2006), Galdi and Kyed (2010), San Martin, Takahashi and Tucsnak (2007)