A Virtual Element Method for Elasticity

L. Beirão da Veiga
Department of Mathematics
University of Milan

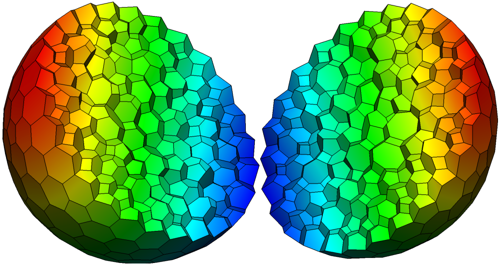
in collaboration with:

F. Brezzi and D. Marini

LISBON, December 2014

The Virtual Element Method

The Virtual Element Method (VEM) is a generalization of the Finite Element Method that takes inspiration from modern Mimetic Finite Difference schemes.



- VEM allows to use very general polygonal and polyhedral meshes, also for high polynomial degrees and guaranteeing the patch test.
- In addition, it may lead to additional advantages, such as “easy to code” highly regular spaces.

Why polygons/polyhedrons?

The interest (and use in commercial codes*) for polygons/polyhedra is recently growing.

- Immediate combination of tets and hexahedrons
- Easier/better meshing of domain (and data) features
- Automatic inclusion of “hanging nodes”
- Adaptivity: more efficient mesh refinement/coarsening
- Generate meshes with more local rotational simmetries
- Robustness to distortion
- ....

$\star$ for example CD-ADAPCO and ANSYS.

Some polytopal methods

- Mimetic F.D. Shashkov, Lipnikov, Brezzi, Manzini, BdV, ...
- HMM: Eymard, Droniou, ...
- Polygonal FEM: Sukumar, Paulino, ...
- Weak Galerkin FEM: Wang, ...
- HHO: Ern, di Pietro
- Polygonal DG: Cangiani, Houston, Georgoulis, ...
- VEM: this talk !!!
- ....

The linear elasticity problem

We consider the linear elasticity problem in two dimensions:

$$\begin{cases} -\mathbf{div}\Big(2\mu\,\boldsymbol{\varepsilon}(\boldsymbol{u}) + \lambda\,\mathrm{div}(\boldsymbol{u})\,\mathcal{I}\Big) = \boldsymbol{f} \quad \text{in } \Omega \\ \boldsymbol{u} = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where

- $\Omega \subset \mathrm{R}^2$ is a polygonal domain;
- λ and μ are the Lamé coefficients in $\mathbb{R}+$;
- the loading f is assumed component-wise in $L^2(\Omega)$.

NOTE: When the ratio $\lambda/\mu \gg 1$ we fall into the range of almost-incompressible materials ...

Variational formulation

The variational formulation of the problem reads:

$$\begin{cases} \text{Find } \boldsymbol{u} \in \mathbf{V} \text{ such that} \\ a(\boldsymbol{u}, \boldsymbol{v}) =< \boldsymbol{f}, \boldsymbol{v} > \quad \forall\, \boldsymbol{v} \in \mathbf{V} \end{cases}$$

where the space

$$\mathbf{V} := (H^1_0(\Omega))^2$$

and the bilinear forms

$$a(\boldsymbol{v}, \boldsymbol{w}) := 2\mu \int_\Omega \varepsilon(\boldsymbol{v}) : \varepsilon(\boldsymbol{w})\, \mathrm{d}\mathbf{x} + \lambda \int_\Omega \mathrm{div}\, \boldsymbol{v}\, \mathrm{div}\, \boldsymbol{w}\, \mathrm{d}\mathbf{x}$$
$$=: 2\mu\, a_\mu(\boldsymbol{v}, \boldsymbol{w}) + \lambda\, a_\lambda(\boldsymbol{v}, \boldsymbol{w}).$$

NOTE: for given μ, λ the form $a(\cdot, \cdot)$ is symmetric, continuous and coercive on $\mathbf{V}$. The problem above is well posed.

A Virtual Element Method

We will build a discrete problem in following form

$$\begin{cases} \text{find } \boldsymbol{u}_h \in \mathbf{V}_h \text{ such that} \\ a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = < \boldsymbol{f}_h, \boldsymbol{v}_h > \qquad \forall\, \boldsymbol{v}_h \in \mathbf{V}_h, \end{cases}$$

where

- $\mathbf{V}_h \subset \mathbf{V}$ is a finite dimensional space;
- $a_h(\cdot,\cdot)\ :\ \mathbf{V}_h \times \mathbf{V}_h \to \mathbb{R}$ is a discrete bilinear form approximating the continuous form $a(\cdot,\cdot)$;
- $< \boldsymbol{f}_h, \boldsymbol{v}_h >$ is a right hand side term approximating the load.

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Let $k \geq 1$ be a fixed integer index. Such index will represent the degree of accuracy of the method.

The local spaces $\mathbf{V}_{h|E}$

Let $\mathcal{T}_h$ be a simple polygonal mesh on Ω. This can be any decomposition of Ω in non overlapping polygons E with straight faces.

The space V_h will be defined element-wise, by introducing

- local spaces $\mathbf{V}_{h|E}$;
- the associated local degrees of freedom.

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For all $E \in \mathcal{T}_h$:

$$\mathbf{V}_{h|E} = \Big\{ \boldsymbol{v}_h \in [H^1(E) \cap C^0(E)]^2 \ : \ -\Delta \boldsymbol{v}_h \in [\mathrm{P}_{k-2}(E)]^2, \\ \boldsymbol{v}_h|_e \in [\mathrm{P}_k(e)]^2 \quad \forall e \in \partial E \Big\}.$$

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- the functions $\boldsymbol{v}_h \in \mathbf{V}_{h|E}$ are known explicitly on ∂E;
- the functions $\boldsymbol{v}_h \in \mathbf{V}_{h|E}$ are unknown inside the element E!
- it holds $[\mathrm{P}_k(E)]^2 \subseteq \mathbf{V}_{h|E}$

Degrees of freedom for $\mathbf{V}_{h|E}$

Let $E \in \mathcal{T}_h$ and $\boldsymbol{v}_h \in \mathbf{V}_{h|E}$.

- pointwise values $\boldsymbol{v}_h(\nu)$ at all corners ν of E
- $(k-1)$ pointwise values on each edge:

$$\boldsymbol{v}_h(x_i^e)\ , \quad \{x_i^e\}_{i=1}^{k-1} \text{ distinct points on edge } e\,;$$

- volume moments:

$$\int_E \boldsymbol{v}_h \cdot \boldsymbol{p}_{k-2} \qquad \forall \boldsymbol{p}_{k-2} \in [\mathrm{P}_{k-2}]^2(E)$$

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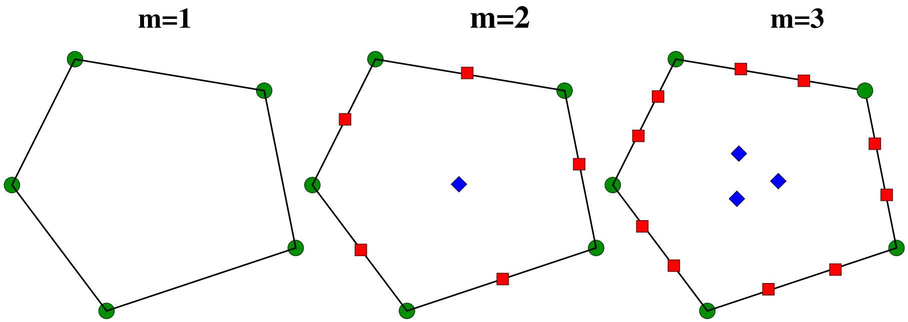
The dimension of the space $\mathbf{V}_{h|E}$ is

$$\dim(\mathbf{V}_{h|E}) = 2Nk + k(k-1),$$

with N number of edges of E.

Depiction of the degrees of freedom for $\mathbf{V}_{h|E}$



Green dots stand for vertex pointwise values

Red squares represent pointwise values on edges

Blue squares represent internal (volume) moments

The space V_h and the form $a_h(\cdot, \cdot)$

The global space $\mathbf{V}_h$ is built by assembling the local spaces $\mathbf{V}_{h|E}$ as usual:

$$\mathbf{V}_h = \{\boldsymbol{v} \in \mathbf{V} \ : \ \boldsymbol{v}|_E \in \mathbf{V}_{h|E} \, \forall E \in \mathcal{T}_h\}.$$

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$$\mathbf{V}_h = \{\boldsymbol{v} \in \mathbf{V} \;:\; \boldsymbol{v}|_E \in \mathbf{V}_{h|E} \,\forall E \in \mathcal{T}_h\}.$$

The bilinear form $a_h(\cdot,\cdot)$ is built element by element

$$a_h(\boldsymbol{v}_h, \boldsymbol{w}_h) = \sum_{E\in\mathcal{T}_h} a_h^E(\boldsymbol{v}_h, \boldsymbol{w}_h) \quad \forall\, \boldsymbol{v}_h,\, \boldsymbol{w}_h \in V_h,$$

where

$$a_h^E(\cdot,\cdot) \;:\; \mathbf{V}_{h|E} \times \mathbf{V}_{h|E} \longrightarrow \mathbf{R}$$

are symmetric bilinear forms that approximate

$$a_h^E(\cdot,\cdot) \;\simeq\; a(\cdot,\cdot)|_E.$$

Building the local bilinear forms

We recall the exact local form

$$a^E(\boldsymbol{v}, \boldsymbol{w}) := 2\mu \int_E \varepsilon(\boldsymbol{v}) : \varepsilon(\boldsymbol{w}) \; + \; \lambda \int_E \mathrm{div}(\boldsymbol{v})\,\mathrm{div}(\boldsymbol{w})$$
$$=: \quad 2\mu\, a^E_\mu(\boldsymbol{v}, \boldsymbol{w}) \quad + \quad \lambda\, a^E_\lambda(\boldsymbol{v}, \boldsymbol{w}).$$

Analogously we will introduce

$$a^E_h(\boldsymbol{v}_h, \boldsymbol{w}_h) := 2\mu\, a^E_{h,\mu}(\boldsymbol{v}_h, \boldsymbol{w}_h) + \lambda\, a^E_{h,\lambda}(\boldsymbol{v}_h, \boldsymbol{w}_h),$$

for all $\boldsymbol{v}_h, \boldsymbol{w}_h$ in $\mathbf{V}_{h|E}$.

NOTE: we do not have a completely explicit expression for the functions in $\mathbf{V}_{h|E}$!

Bilinear form $a^E_{h,\lambda}(\cdot,\cdot)$

Note that $\forall \boldsymbol{v}_h \in \mathbf{V}_{h|E}$ and $\forall p \in \mathrm{P}_{k-1}(E)$ we can compute

$$\int_E \mathrm{div}(\boldsymbol{v}_h)\, p \; = - \int_E \boldsymbol{v}_h \cdot \boldsymbol{\nabla} p + \int_{\partial E} (\boldsymbol{v}_h \cdot \mathbf{n}_E) p \, .$$

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Therefore we can compute the L^2 projection of $\mathrm{div}(\boldsymbol{v}_h)$ on $\mathrm{P}_{k-1}(E)$:

$$\begin{cases} \Pi^0_{k-1}\big(\mathrm{div}(\boldsymbol{v}_h)\big) \in \mathrm{P}_{k-1}(E) \\ \displaystyle\int_E \big(\Pi^0_{k-1}\mathrm{div}(\boldsymbol{v}_h)\big)\, p = \int_E \mathrm{div}(\boldsymbol{v}_h)\, p \quad \forall p \in \mathrm{P}_{k-1}(E). \end{cases}$$

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We simply define, for all $\boldsymbol{v}_h, \boldsymbol{w}_h \in \mathbf{V}_{h|E}$,

$$a^E_{h,\lambda}(\boldsymbol{v}_h, \boldsymbol{w}_h) := \int_E \Pi^0_{k-1}\big(\mathrm{div}(\boldsymbol{v}_h)\big)\, \Pi^0_{k-1}\big(\mathrm{div}(\boldsymbol{w}_h)\big).$$

Bilinear form $a^E_{h,\mu}(\cdot,\cdot)$

Note that $\forall \boldsymbol{v}_h \in \mathbf{V}_{h|E}$ and $\forall \boldsymbol{p} \in [\mathrm{P}_k(E)]^2$ we can compute

$$a^E_\mu(\boldsymbol{v}_h, \boldsymbol{p}) = \int_E \varepsilon(\boldsymbol{v}_h) : \varepsilon(\boldsymbol{p}) = -\int_E \boldsymbol{v}_h \cdot \mathbf{div}\big(\varepsilon(\boldsymbol{p})\big) + \int_{\partial E} \boldsymbol{v}_h \cdot \big(\varepsilon(\boldsymbol{p})\mathbf{n}_E\big) \ .$$

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Thus we can compute the "a_μ" projection of $\boldsymbol{v}_h$ on $\mathrm{P}_k(E)$:

$$\begin{cases} \Pi^a_k(\boldsymbol{v}_h) \in \mathrm{P}_k(E) \\ a^E_\mu(\Pi^a_k(\boldsymbol{v}_h), \boldsymbol{p}) = a^E_\mu(\boldsymbol{v}_h, \boldsymbol{p}) \quad \forall \boldsymbol{p} \in \mathrm{P}_k(E) \\ \text{Fix the kernel part} \end{cases}$$

For instance, we can fix the kernel

$$\text{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -y \\ x \end{pmatrix} \right\}$$

by using an euclidean scalar product on the d.o.f. values.

Bilinear form $a^E_{h,\mu}(\cdot,\cdot)$

It clearly holds, for all $\boldsymbol{v}_h, \boldsymbol{w}_h$ in $\mathbf{V}_{h|E}$:

$$a^E_\mu(\boldsymbol{v}_h, \boldsymbol{w}_h) = a^E_\mu(\Pi^a_k(\boldsymbol{v}_h), \Pi^a_k(\boldsymbol{w}_h)) + a^E_\mu(\boldsymbol{v}_h - \Pi^a_k(\boldsymbol{v}_h), \boldsymbol{w}_h - \Pi^a_k(\boldsymbol{w}_h)) .$$

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We define

$$a^E_{h,\mu}(\boldsymbol{v}_h, \boldsymbol{w}_h) := a^E_\mu(\Pi^a_k(\boldsymbol{v}_h), \Pi^a_k(\boldsymbol{w}_h)) + s^E(\boldsymbol{v}_h - \Pi^a_k(\boldsymbol{v}_h), \boldsymbol{w}_h - \Pi^a_k(\boldsymbol{w}_h))$$

where the bilinear form $s^E(\cdot,\cdot)$ is only required to scale correctly.

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where the bilinear form $s^E(\cdot,\cdot)$ is only required to scale correctly.

For instance, for all $\boldsymbol{v}_h, \boldsymbol{w}_h$ in $\mathbf{V}_{h|E}$,

$$s^E(\boldsymbol{v}_h, \boldsymbol{w}_h) \;=\; 2\mu \sum_{i=1}^{\#\text{dofs}} \text{dof}_i(\boldsymbol{v}_h)\text{dof}_i(\boldsymbol{w}_h).$$

Bilinear form $a^E_{h,\mu}(\cdot,\cdot)$

The bilinear form satisfies the following.

Consistency:

$$a^E_{h,\mu}(\boldsymbol{p},\boldsymbol{v}_h) = a^E_{\mu}(\boldsymbol{p},\boldsymbol{v}_h) \qquad \forall \boldsymbol{v}_h \in \mathbf{V}_{h|E},\ \forall \boldsymbol{p} \in [\mathbb{P}_k(E)]^2 \,.$$

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Stability:

There exist $\alpha_\star, \alpha^\star \in \mathbb{R}+$ such that

$$\alpha_\star a^E_{\mu}(\boldsymbol{v}_h,\boldsymbol{p}) \le a^E_{h,\mu}(\boldsymbol{v}_h,\boldsymbol{p}) \le \alpha^\star a^E_{\mu}(\boldsymbol{v}_h,\boldsymbol{p}) \qquad \forall \boldsymbol{v}_h \in \mathbf{V}_{h|E} .$$

NOTE: in order for $\alpha_\star, \alpha^\star$ to be uniform (element-independent) some mesh regularity assumptions are needed (eg. star-shaped with respect to a ball, minimal edge length condition).

The discrete load term

We consider $k \geq 2$ first. Let, for all $E \in \mathcal{T}_h$, the approximated load $\boldsymbol{f}_h|_E$ be the L_2-projection of $\boldsymbol{f}|_E$ on $\mathrm{P}_{k-2}(E)$.

Then

$$(\boldsymbol{f}_h, \boldsymbol{v}_h)_h := \sum_{E \in \mathcal{T}_h} \int_E \boldsymbol{f}_h \boldsymbol{v}_h$$

that is computable due to the internal dofs of $\mathbf{V}_{h|E}$.

In the case $k = 1$ a simple integration rule based on the vertex values of the polygon can be used, for instance

$$(\boldsymbol{f}_h, \boldsymbol{v}_h)_h := \sum_{E \in \mathcal{T}_h} \left(\int_E \boldsymbol{f} \right) \frac{1}{N_E} \sum_{\nu \in \partial E} \boldsymbol{v}_h(\nu).$$

Note: for $k = 2$ better choices can be made.

A λ−uniform convergence result

Mesh assumptions: let every element E be star-shaped with respect to a ball of radius $\geq \rho h_E$, and let every two vertexes be at least ρh_E distance apart (ρ fixed for the whole mesh family).

Theorem: [BdV, Brezzi, Marini, 2013]

It holds the error estimate

$$||\boldsymbol{u} - \boldsymbol{u}_h||_{H^1(\Omega)} \leq C h^s |\boldsymbol{u}|_{H^{s+1}(\Omega)} , \quad 1 \leq s \leq k,$$

with C independent of λ if $k \geq 2$.

NOTE: and inf-sup condition holds for $k \geq 2$, granting the independence from λ. In the case $k = 1$ there exists a class of meshes for which the inf-sup holds (see [BdV, Lipnikov, 2010]).

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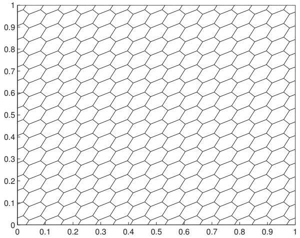
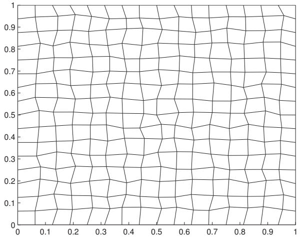
NOTE: it immediately extends to the Stokes problem!

Low order numerical tests

We consider only $k = 1$ and two family of meshes on $\Omega = [0, 1]^2$:



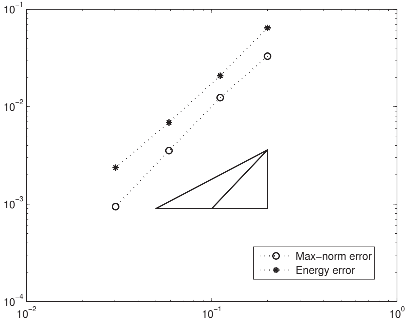


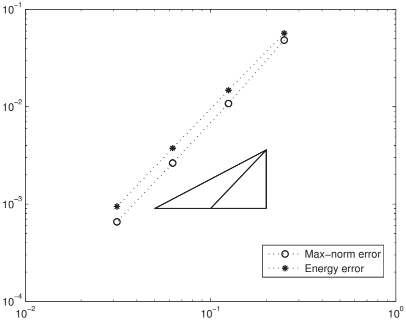
We set $\lambda = \mu = 1$ and the loading $\boldsymbol{f}$ in accordance with the solution

$$\boldsymbol{u}(x, y) = \big(\sin(\pi x)\sin(\pi y)\,,\ \sin(\pi x)\sin(\pi y)\big).$$

Low order numerical tests

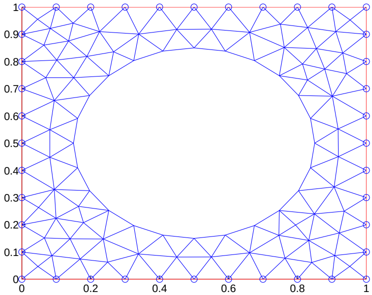
Convergence plots for max-error at nodes and discrete energy error.





An additional example

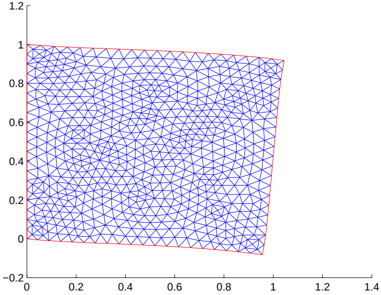
Consider a $[0, 1]^2$ domain of elastic material ($\mu = \lambda = 1$) with a stiff inclusion ($\mu = \lambda = 10^4$).
The domain is clamped on the left side and subjected to a vertical force on the right hand side.

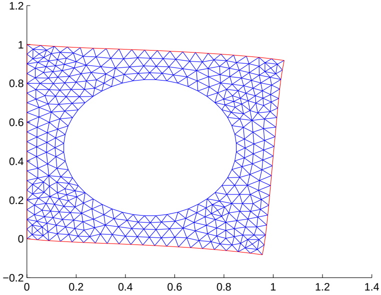


Very little happens in the inclusion, but it cannot be ignored: FEM must mesh the inclusion while VEM can use a single element.

An additional example

Sample of meshes ($F2$ and $V2$) used, after deformation:





Coarser and finer meshes where used ($F1$, $F2$, $F3$ and $V1$, $V2$, $V3$).

An additional example

Vertical displacement at upper right corner: -0.083818847125168
(reference with a finer FEM mesh)

mesh type	dofs	displacement
mesh F1	179	-0.080290340968924
mesh V1	123	-0.080289101597552
mesh F2	673	-0.082401721109781
mesh V2	428	-0.082400518411353
mesh F3	2609	-0.083363990429520
mesh V3	1584	-0.083362795720282

Nonlinear elasticity and inelasticity

Together with C. Lovadina and D. Mora, we are currently developing the VEM method for small deformation problems in structural mechanics.

Without showing the details of the construction, we here present some preliminary test for $k = 1$ and the following models:

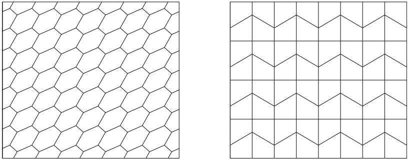
- Henky-Mises nonlinear elasticity
- Von Mises plasticity with linear isotropic and kinematic hardening

NOTE: the method is built in such a way to include (automatically) general “black-box” constitutive algorithms, that are applied only once per element (as in single Gauss point integration).

Henky-Mises elasticity

The lamé constants λ, μ depend on the (point-wise) norm of the strain.

We consider a problem with known regular trigonometric solution on $[0,1]^2$ and two family of meshes:



We report the maximum displacement error at the nodes and the error in an H^1 discrete norm

$$|||\mathbf{v}|||_{1,h}^2 := \sum_{e \in \mathcal{E}_h} h_e \, \|\partial \mathbf{v} / \partial \mathbf{t}_e\|_{0,e}^2$$

The results are as follows:

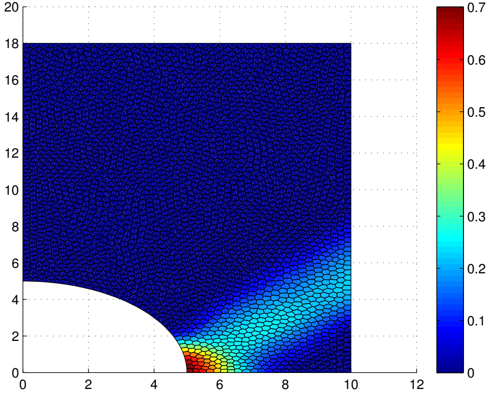
Mesh	$1/h$	N	$E^h_{0,\infty}$	$R_{0,\infty}$	$E^h_{1,2}$	$R_{1,2}$
$\mathcal{T}_h^2$		380	1.2225e-2	–	1.9750e-1	–
		1560	2.2164e-3	2.42	8.4773e-2	1.20
		3292	1.1218e-3	1.82	5.8722e-2	0.98
		5664	6.7588e-4	1.87	4.5076e-2	0.98
		9084	3.9767e-4	2.25	3.4684e-2	1.11
		12816	2.8737e-4	1.89	2.9372e-2	0.97
		17188	2.1964e-4	1.83	2.5529e-2	0.96
		22848	1.5928e-4	2.26	2.1811e-2	1.10
$\mathcal{T}_h^5$	4	50	1.5401e-1	–	1.0516e-0	–
	8	162	3.3021e-2	2.62	5.3972e-1	1.14
	16	578	7.1005e-3	2.42	2.7525e-1	1.06
	32	2178	1.6650e-3	2.19	1.3832e-1	1.04
	64	8450	4.1133e-4	2.06	6.9382e-2	1.02
	128	33282	9.0462e-5	2.21	3.2452e-2	1.05

Von Mises plasticity

We consider the classical stretching test of a plastic strip with a hole, using a return map algorithm to solve the von Mises plasticity model:

$$E = 70, \quad \nu = 0.2, \quad r_0 = 0.8.$$

Plastic parameter γ:



Conclusions

- We presented a Virtual Element Method for linear elasticity:
  - (local) discrete spaces
  - degrees of freedom
  - building the discrete bilinear forms
- The discrete bilinear forms where shown to be k-consistent and stable
- The method was shown to be convergent and free of volumetric locking.
- We have shown first extensions to the nonlinear case.
- Virtual Elements is a very recent technology, that we believe to be full of potential!