Numerical investigation of long range segregation models

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Outlook and Problem Description

- (A) Problem description.
- (B) Basic theorem and facts.
- (C) Numerical schemes.
- (D) Further directions and related problems

(A) Problems description

Two Models in spatial segregation:

- Adjacent segregation: Particles annihilate on contact, common surface of separation.

  Appears in modeling of population density.

  Competition models of Lotka-Volterra type.

(A) Problems description

Two Models in spatial segregation:

- Adjacent segregation: Particles annihilate on contact, common surface of separation.
  Appears in modeling of population density.
  Competition models of Lotka-Volterra type.
- At Distance: Species interact at a distance from each other. More complex geometric problem: Recent work by L. Caffarelli, S. Partrizi, V. Quitalo, [CPQ]

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- The steady-states of m competing components in Ω is given by

$$\begin{cases} -\Delta u_i^\varepsilon = -\frac{1}{\varepsilon} u_i^\varepsilon(x) \sum_{j \neq i}^m \; a_{ij} \left(u_j^\varepsilon(x)\right)^\alpha + f_i(x, u_i^\varepsilon(x)) & \text{in } \Omega \\ u_i > o & \text{in } \Omega \\ u_i(x) = \phi_i(x) & \text{on } \partial\Omega, \end{cases}$$

where $\alpha = 1, 2$.

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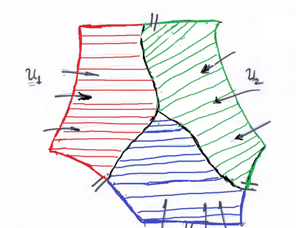
- The boundary values ϕ_i are non-negative and have disjoint supports on the boundary, i.e,

$$\phi_i \cdot \phi_j = 0 \quad \text{on} \quad \partial\Omega.$$

The goal: study the system as $\varepsilon \to 0$ in model 1

- With out loos of generality assume $a_{ij} = 1, f_i(x, u_i) = 0$.
  Theorem1[CTV]:
  Let $U^\varepsilon = (u_1^\varepsilon, ..., u_m^\varepsilon)$ be a solution of system at fixed ε. Let $\varepsilon \to 0$, then there exists $U \in (H^1(\Omega))^m$ such that for all $i = 1, ..., m$:
  1. up to a subsequences, $u_i^\varepsilon \to u_i$ strongly in $H^1(\Omega)$,
  2. $u_i \cdot u_j = 0$ if $i \neq j$ a.e in Ω,
  3. $\Delta u_i = 0$ in the set $\{u_i > 0\}$.
  4. Let x belongs to interface such that $m(x) = 2$ then $\lim_{y \to x} \nabla u_i(y) = -\lim_{y \to x} \nabla u_j(y)$.



Asymptotic behaviour as $\varepsilon \to 0$ in model 1

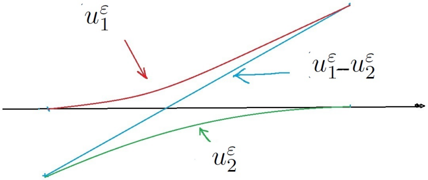
The limiting solution belong to the following class:

$$S = \{U = (u_1, \cdots, u_m) \in (H^1(\Omega))^m : u_i \geq 0, u_i \cdot u_j = 0 \text{ } if \text{ } i \neq j$$

$$u_i = \phi_i \text{ on } \partial\Omega, \; -\Delta u_i \leq 0 \;, -\Delta(u_i - \sum_{j \neq i} u_j) \geq 0\}.$$

The Limit of system in the case $d = 1$, $m = 2$:

$$\begin{cases} \Delta u_1^\varepsilon = \frac{1}{\varepsilon} u_1^\varepsilon(x) u_2^\varepsilon(x) & \text{in } \Omega \\ \Delta u_2^\varepsilon = \frac{1}{\varepsilon} u_2^\varepsilon(x) u_1^\varepsilon(x) & \text{in } \Omega \end{cases}$$



The Limit of system in the case $m = 2$

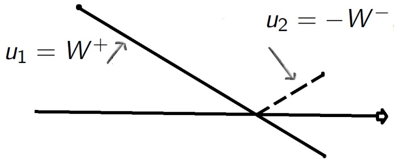
Theorem 2[CTV]: Let W be harmonic with the boundary data $\phi_1 - \phi_2$. Let $u_1 = W^+, u_2 = -W^-$, then the pair(u_1, u_2) is the limit configuration of any sequences $(u_1^\varepsilon, u_2^\varepsilon)$

$$\| u_i^\varepsilon - u_i \|_{H^1(\Omega)} \leq C(\varepsilon)^{1/6} \quad as\ \varepsilon \to 0, \quad i = 1, 2.$$

Remark: The two-phases free boundary (talk of Rodrigues)

$$\Delta u = \lambda^+ \chi_{\{u>0\}} - \lambda^- \chi_{\{u<0\}}$$

is spacial case with $u_1 = u^+, u_2 = -u^-, f_1 = \lambda^+, f_2 = \lambda^-$.

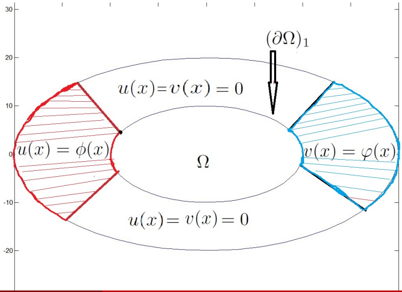


Segregation at distance

- The system has some similarity with previous model, however the annihilation of the coefficient for u_1 at the point x is not $u_2(x)$ any longer, but involves the values of u_2 in a full neighborhood of the point x. Thus we need to prescribe u_1 and u_2 in a neighborhood of Ω.

Segregation at distance

- The system has some similarity with previous model, however the annihilation of the coefficient for u_1 at the point x is not $u_2(x)$ any longer, but involves the values of u_2 in a full neighborhood of the point x. Thus we need to prescribe u_1 and u_2 in a neighborhood of Ω.
- Denote $(\partial\Omega)_1 := \{x \in \Omega^c : d(x, \Omega) \leq 1\}$.
- The solution of the first model can be used as initial guess in second model.



The Model of segregation at distance

The model is described by the following system

$$\begin{cases} -\Delta u_i^\varepsilon = -\frac{1}{\varepsilon} u_i^\varepsilon \sum_{i\neq j} H(u_j^\varepsilon)(x) & \text{in } \Omega, \\ u_i(x) = \phi_i(x) & \text{in } (\partial\Omega)_1, \\ i = 1 \cdots m. \end{cases} \tag{1}$$

where

$$H(u_j^\varepsilon)(x) = \int_{B_1(x)} u_j^\varepsilon(y)dy$$

or

$$H(u_j^\varepsilon)(x) = \sup_{y\in B_1(x)} u_j^\varepsilon(y).$$

Assumptions: $\phi_i(x)$ for $i = 1, \cdots, m$ are non-negative C^1 functions such that have disjoint supports in distance more than one

$$(supp\, \phi_i(x))_1 \cap (supp\, \phi_j(x))_1 = \emptyset.$$

Existence and Uniqueness

Lemma

For each $\varepsilon > 0$, there exist a unique positive solution $(u_1^\epsilon, \cdots, u_m^\epsilon)$ of system in (1).

Sketch of the Proof

- Consider the harmonic extension u_i^0 for $i = 1, \cdots, m$ given by

$$\begin{cases} -\Delta u_i^0 = 0 & \text{in } \Omega, \\ u_i^0 = \phi_i & \text{on } \partial\Omega, \end{cases} \tag{2}$$

- Given u_i^k consider the solution of the following linear system

$$\begin{cases} \Delta u_i^{k+1} = \frac{1}{\varepsilon} u_i^{k+1} \sum_{i \neq j} H(u_j^k)(x) & \text{in } \Omega, \\ u_i^{k+1}(x) = \phi_i(x) & \text{on } (\partial\Omega)_1, \end{cases} \tag{3}$$

Sketch of the Proof for Existence and Uniqueness

- The following inequalities hold:

$$u_i^0 \geq u_i^2 \cdots \geq u_i^{2k} \geq \ldots u_i^{2k+1} \geq \cdots u_i^3 \geq u_i^1, \quad \text{in}\,\Omega.$$

$$u_i^{2k} \to u_i^{\star} \quad \text{uniformly in } \Omega$$
$$u_i^{2k+1} \to u_i^{\diamond} \quad \text{uniformly in } \Omega$$

- Next we show that

$$u_i^{\star} = u_i^{\diamond}$$

- Assume there exist another solution $(w_1, \cdots, w_n)$ of (1), then
- We will prove that the following hold:

$$u_i^{2k+1} \leq w_i \leq u_i^{2k}, \qquad \text{for } \; k \geq 0, \tag{4}$$

which shows

$$u_i = w_i.$$

Basic Estimates

For simplicity assume $m = 2$:

$$\begin{cases} -\Delta u^{\varepsilon}(x) = -\frac{u^{\varepsilon}(x)}{\varepsilon} \int_{B_1(x)} v^{\varepsilon}(y)dy & \text{in } \Omega, \\ -\Delta v^{\varepsilon}(x) = -\frac{v^{\varepsilon}(x)}{\varepsilon} \int_{B_1(x)} u^{\varepsilon}(y)dy & \text{in } \Omega, \\ u(x) = \phi(x) & \text{in } (\partial\Omega)_1, \\ v(x) = \varphi(x) & \text{in } (\partial\Omega)_1. \end{cases} \tag{5}$$

Let $(u^{\varepsilon}, v^{\varepsilon})$ be a solution of system (5). There exist constant C_1, C_2 independent of ε such that if $(u^{\varepsilon}, v^{\varepsilon})$ be a solution of system (5) then

- $\int_{\Omega} u^{\varepsilon}(\int_{B_1(x)} v^{\varepsilon}(y)\, dy)\, dx \leq C_1\varepsilon,$
- $\int_{\Omega} |\nabla u^{\varepsilon}|^2\, dx \leq C_2,$
- $\int_{\Omega} |\nabla v^{\varepsilon}|^2\, dx \leq C_2,$
- As ε tends to zero there exist subsequences $\{u^{\varepsilon_j}\}$ and $\{v^{\varepsilon_j}\}$ and non-negative u, v such that

$$u^{\varepsilon_j} \to u \quad \text{in } W^{1,2}, \qquad v^{\varepsilon_j} \to v \quad \text{in } W^{1,2}.$$

Qualitative Properties

Theorem (L. Caffarelli, S. Patrizi, and V. Quitalo)

Let u and v be the limiting solutions as ε tends to zero. Then

- *u and v are locally Lipschitz continuous.*
- *The free boundaries $\Gamma_1 = \partial\{x \in \Omega : u(x) > 0\}$, $\Gamma_2 = \partial\{x \in \Omega : v(x) > 0\}$, have distance one from each other.*
- *In 2-dimensions the free boundaries Γ_1, Γ_2 are C^1 curves.*
- *The functions u and v are harmonic in their supports.*

The Laplacians $\Delta u, \Delta v$, are jump measures along Γ_1, Γ_2,

$$\Delta u = u_\nu H^{n-1} \mid_{\Gamma_1} \qquad \Delta v = v_\nu H^{n-1} \mid_{\Gamma_2} \text{ in } B_r \text{ in distributional sense.}$$

Assume $0 \in \Gamma_1$, also let e_2 be exterior normal derivative at 0. We obtain a corresponding point in Γ_2 which has distance one from 0.

Qualitative Properties

Theorem (L. Caffarelli, S. Patrizi, and V. Quitalo)

- *Let* $D_h = B_h(0) \cap \{x : d(x, \Gamma_1) \leq h^2\}$ *for a small fixed* h. *Let* E_h *be the image of* D_h *through* $y = x + \nu(x)$ *with* $x \in D_h$. *Then,*

$$\int_{D_h} \Delta u \, dx = \int_{E_h} \Delta v \, dx$$

$$\frac{u_\nu(0)}{v_\nu(e_2)} = \frac{\kappa(0)}{\kappa(e_2)},$$

where $\kappa(x)$: *mean curvature.*

- *Let* $\Gamma_1^h = \Gamma_1 \cap B_h(0)$, *and* $\Gamma_2^h = \{x + \nu(x) : x \in \Gamma_1^h\}$. *Then as* $h \to 0$ *we have*

$$\frac{\int_{\Gamma_2^h} 1 dA}{\int_{\Gamma_2^h} 1 dA} \to \frac{\kappa(0)}{\kappa(e_2)}$$

Free Boundary Condition in dimension one

$$\begin{cases} u_\varepsilon''(x) = \frac{u_\varepsilon(x)}{\epsilon} \sup_{y\in(x-1,x+1)} v_\varepsilon(y) & \text{in } (-a,a), \\ v_\varepsilon''(x) = \frac{v_\varepsilon(x)}{\varepsilon} \sup_{y\in(x-1,x+1)} u^\varepsilon(y) & \text{in } (-a,a), \\ u(x) = \phi(x) & \text{in } (-a-1,-a), \\ v(x) = \varphi(x) & \text{in } (a,a+1). \end{cases} \tag{6}$$

We have

$$\sup_{y\in(x-1,x+1)} v_\varepsilon(y) = v_\varepsilon(x+1)$$

$$v_\varepsilon''(x+1) = \frac{v_\varepsilon(x+1)}{\varepsilon} \sup_{y\in(x,x+2)} u_\varepsilon(y) = \frac{v_\varepsilon(x+1)}{\varepsilon} u_\varepsilon(x)$$

This shows for every ε

$$(u_\varepsilon(x) - v_\varepsilon(x+1))'' = 0.$$

$$\Rightarrow (u(x) - v(x+1))'' = 0, \quad \text{and} \quad (v(x) - u(x-1))'' = 0.$$

The free Boundary condition in dimension two

Lemma

Let u and v be the limiting solutions as ε tends to zero. Then

$$\begin{cases} -\Delta(u(x) - v(x - \frac{\nabla u(x)}{|\nabla u(x)|}) = 0 \quad in\{u > 0\}, \\ -\Delta(v(x) - u(x - \frac{\nabla v(x)}{|\nabla v(x)|}) = 0 \quad in\{v > 0\}. \end{cases} \tag{7}$$

Approximation for Model 1

For simplicity assume that $n = 1, m = 2$. We use the facts $u - v$ is harmonic in $\Omega, u \cdot v = 0, u, v \geq 0$. Using finite difference

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} - \frac{v_{i+1} - 2v_i + v_{i-1}}{h^2} = 0$$

Let $\overline{u}_i = \frac{u_{i+1}+u_{i-1}}{2}, \quad \overline{v}_i = \frac{v_{i+1}+v_{i-1}}{2}$. Imposing the conditions $u_i \cdot v_i = 0, u_i \geq 0$ and $v_i \geq 0$, we will obtain u_i and v_i by the following formula

$$\text{For} \quad i = 2 \cdots n-1$$

$$\begin{cases} u_i^{(k+1)} = \max\left(\frac{u_{i+1}^{(k)}+u_{i-1}^{(k)}}{2} - \frac{v_{i+1}^{(k)}+v_{i-1}^{(k)}}{2}, 0\right) = \max\left(\overline{u}_i^k - \overline{v}_i^k, 0\right) \\ v_i^{(k+1)} = \max\left(\frac{v_{i+1}^{(k)}+v_{i-1}^{(k)}}{2} - \frac{u_{i+1}^{(k)}+u_{i-1}^{(k)}}{2}, 0\right) = \max\left(\overline{v}_i^k - \overline{u}_i^k, 0\right) \end{cases}$$

Iterative Method for second Model in dimension one

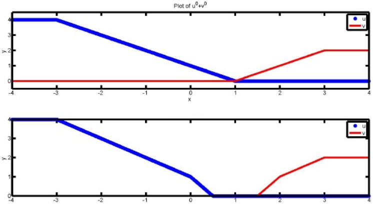
- Let $n = 1, \Omega = (-a, +a)$. The limiting u and v satisfy

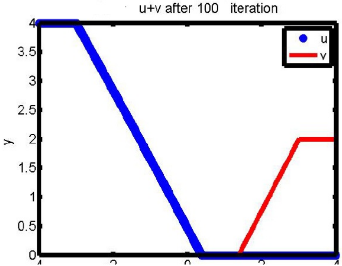
$$\begin{cases} (u(x) - v(x+1))'' = 0, \\ (v(x) - u(x-1))'' = 0. \end{cases}$$

- Discretize Ω by $-a = x_0, x_1 = x_0 + h, \cdots x_N = +a$ be of $\Omega = (-a, +a)$ with L be such that $L \cdot h = 1$.
- Let W be harmonic with $\phi - \varphi$ as boundary data. Set $u^0 = W^+$, $v^0 = W^-$.
- Update u_i and v_i by

$$\begin{cases} u^{(k+1)}(i) = \max\left(\overline{u}_i^{(k)} - \overline{v}_{i+L}^{(k)}, 0\right) \\ v^{(k+1)}(i) = \max\left(\overline{v}_i^{(k)} - \overline{u}_{i+L}^{(k)}, 0\right) \end{cases}$$

Iterative Method

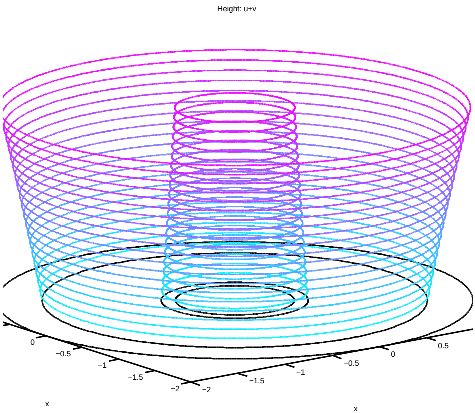




Examples

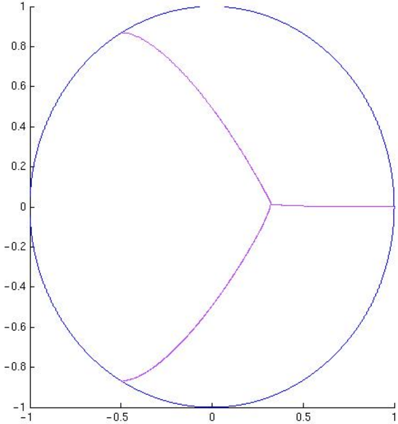
- The $\Omega = B_2(0) \setminus B_{.5}(0)$. The boundary values are

$$u = 1 \quad \text{on } \partial B_{.5}(0) \quad v = 1 \quad \text{on } \partial B_2(0),$$



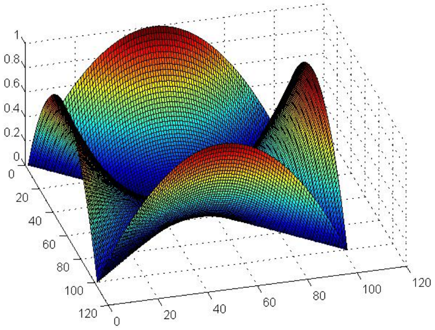
Examples

- The below figure is the case that $\Omega = B_1$ and $\phi_1(x) = \phi_1(x) = |sin(\frac{3}{2}\Theta)|, \phi_3(x) = |3sin(\frac{3}{2}\Theta)|$ with the same process we got the interfaces after 8 iteration.



Example

- we applied second method with $\Omega = [0, 1] \times [0, 1]$ $, \phi_1 = 1 - x^2, \phi_2 = 1 - y^2, \phi_3 = 1 - x^2, \phi_4 = 1 - y^2$



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Thank you, Questions?