On the partial regularity of suitable weak solutions: Pointwise estimates and space-time decay

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Workshop on PDE's and Biomedical Applications

December 4-6, 2014, Lisbon, Portugal

The subject of the talk is part of two papers with P. Maremonti:

F. C. and P. Maremonti, *A remark on the partial regularity of a suitable weak solution to the Navier-Stokes Cauchy problem*, submitted.

F. C. and P. Maremonti, *On the spatial asymptotic decay of a suitable weak solution to the Navier-Stokes Cauchy problem*, submitted.

Introduction

We study the Navier-Stokes Cauchy problem:

$$\begin{aligned} &v_t + v \cdot \nabla v + \nabla \pi_v = \Delta v, \;\; \nabla \cdot v = 0, \;\text{ in } (0,T) \times \mathbb{R}^3, \\ &v(0,x) = v_\circ(x) \text{ on } \{0\} \times \mathbb{R}^3. \end{aligned} \tag{1}$$

It is well known that in

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the following existence result is proved:

Theorem 1 (Here $J^2(\mathbb{R}^3)$:=completion of $\mathscr{C}_0(\mathbb{R}^3)$ in L^2-norm)

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Definition 1 (Here $J^{1,2}(\mathbb{R}^3)$:=completion of $\mathscr{C}_0(\mathbb{R}^3)$ in $W^{1,2}(\mathbb{R}^3)$)

A pair (v, π_v) *is said a suitable weak solution to problem* (1) *if*

i) *for all* $T > 0$, $v \in L^2(0, T; J^{1,2}(\mathbb{R}^3))$ *and* $\pi_v \in L^{\frac{5}{3}}((0, T) \times \mathbb{R}^3)$,

$$\|v(t)\|_2^2 + 2\int_0^t \|\nabla v(\tau)\|_2^2 d\tau \le \|v_\circ\|_2^2, \; t > 0,$$

ii) $\lim_{t \to 0} \|v(t) - v_\circ\|_2 = 0,$

iii) *for all* $t, s \in (0, T)$ *and for all* $\varphi \in C_0^1([0, T) \times \mathbb{R}^3)$:

$$\int_s^t \Big[(v, \varphi_\tau) - (\nabla v, \nabla \varphi) + (v \cdot \nabla \varphi, v) + (\pi_v, \nabla \cdot \varphi)\Big] d\tau + (v(s), \varphi(s)) = (v(t), \varphi(t));$$

iv) *for all* $t \ge s$, *for* $s = 0$ *and a.e. in* $s \ge 0$, *and for all nonnegative* $\phi \in C_0^\infty(\mathbb{R} \times \mathbb{R}^3)$:

$$\int_{\mathbb{R}^3} |v(t)|^2 \phi(t) + 2\int_s^t \int_{\mathbb{R}^3} |\nabla v|^2 \phi \le \int_{\mathbb{R}^3} |v(s)|^2 \phi(s) dx$$

$$+ \int_s^t \int_{\mathbb{R}^3} |v|^2 (\phi_\tau + \Delta \phi) dx d\tau + \int_s^t \int_{\mathbb{R}^3} (|v|^2 + 2\pi_v) v \cdot \nabla \phi dx d\tau.$$

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Following [CKN82] we set

$$M(r) := r^{-2} \iint_{Q_r} (|v|^3 + |v||\pi_v|) dy d\tau + r^{-\frac{13}{4}} \int_{t-r^2}^{t} \Big(\int_{|x-y|<r} |\pi_v| dy \Big)^{\frac{5}{4}} d\tau \,,$$

where $r \in (0, t^{\frac{1}{2}})$, and

$$Q_r(t,x) := \{(\tau, y) : t - r^2 < \tau < t \text{ and } |y - x| < r\},$$

$$Q_r^*(t,x) := \{(\tau, y) : t - \tfrac{7}{8} r^2 < \tau < t + \tfrac{1}{8} r^2 \text{ and } |y - x| < r\}.$$

In [CKN82] it is proved:

Proposition 1

Let (v, π_v) be a suitable weak solution in some parabolic cylinder $Q_r(t,x)$. There exist $\varepsilon_1 > 0$ and $c_0 > 0$ independent of (v, π_v) such that, if $M(r) \le \varepsilon_1$, then

$$|v(\tau, y)| \le c_1^{\frac{1}{2}} r^{-1}, \text{ a. e. in } (\tau, y) \in Q_{\frac{r}{2}}(t,x), \tag{2}$$

where $c_1 := c_0 \varepsilon_1^{\frac{2}{3}}$. In particular, v is regular in $Q_{\frac{r}{2}}(t,x)$.

Proposition 2

There exists a constant $\varepsilon_3 > 0$ with the following property. If (v, π_v) is a suitable weak solution in some parabolic cylinder $Q_r^*(t,x)$ and

$$\limsup_{r \to 0} r^{-1} \iint_{Q_r^*} |\nabla v|^2 dy d\tau \le \varepsilon_3 \,,$$

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In [CKN82], the above propositions are crucial to deduce a partial regularity for a suitable weak solution. However in both the results no dependence on the initial data is given.

Also in the case of their applications, in particular Theorem C and Theorem D, and its Corollary, the results of regularity express on the geometry of regular points, but a behavior in a neighborhood of $t = 0$ does not seem an immediate consequence.

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A first result and comments

We define

$$\|v_\circ\|_{2,w(x)} := \Big(\int\limits_{\mathbb{R}^3} \frac{|v_\circ(y)|^2}{|x-y|} dy \Big)^{\frac{1}{2}}, \quad \mathscr{E}_0(x_0, R_0) := \operatorname*{ess\,sup}_{B(x_0,R_0)} \|v_\circ\|_{2,w(x)} \tag{3}$$

Theorem 2

Let (v, π_v) be a suitable weak solution corresponding to $v_\circ \in J^2(\mathbb{R}^3)$. There exist absolute constants ε_1, C_1 and C_2 such that if

$$C_1 \mathscr{E}_0(x_0, R_0) < 1 \ \text{ and } \ C_2(\mathscr{E}_0^3 + \mathscr{E}_0^{\frac{5}{2}}) \le \varepsilon_1, \tag{4}$$

then

$$|v(t,x)| \le c(\mathscr{E}_0^3 + \mathscr{E}_0^{\frac{5}{2}})^{\frac{1}{3}} t^{-\frac{1}{2}}, \tag{5}$$

provided that (t, x) is a Lebesgue point with $\|v_\circ\|_{2,w(x)} < \infty$ and $x \in B(x_0, R_0)$.

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We remark that for all $v_\circ \in J^2(\mathbb{R}^3)$, by virtue of the Hardy-Littlewood-Sobolev theorem, we have that $\|v_\circ\|_{2,w(x)} < \infty$ almost everywhere in $x \in \mathbb{R}^3$.

Hence, a data in L^2 inherently defines functional $(3)_1$.

But in condition (4) there is more, that is the smallness of the norm $\|\|v_\circ\|_{2,w(x)}\|_{L^\infty(B(x_0,R_0))}$.

We can find several sufficient conditions on $v_\circ \in J^2(\mathbb{R}^3)$ such that assumption (4) is verified. For example, if we consider the assumption made in [CKN82] (in the Corollary), that is $v_\circ \in W^{1,2}(\mathbb{R}^3 \setminus B_R)$, then $\|v_\circ\|_{2,w(x)}$ is a continuous function of x that we can make small outside a suitable ball of radius $R' \geq R$.

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The proof of Theorem 2 is based on Proposition 1.

Also in [CKN82] the regularity result is based on Proposition 1, but our approach to the regularity via the Proposition 1 is different. Indeed, we prove a weighted energy relation (the norm is $\|v(t)\|_{2,w(x)}$) which holds for $t > 0$, provided that $(4)_1$ holds:

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Another feature that expresses the local character of the result is the following. The pointwise behavior of our solution v is given in a neighborhood of $(0, x_0)$, and, further it is as the one of the solutions to the Stokes problem. As far as we know, this property is new.

For this reason our result improves the previous one (that is $\nabla v_\circ \in L^2(\mathbb{R}^3 \setminus B_R)$) proved in [CKN82], in the sense that our thesis gives a bound and a behavior in neighborhood of $t = 0$:

$$|v(t,x)| \leq c(v_\circ)t^{-\frac{1}{2}}, \ (t,x) \in (0,\infty) \times \mathbb{R}^3 \setminus B_{R'}.$$

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Of course estimate (5) also gives a pointwise asymptotic behavior of the solution for large t. However such a behavior is not optimal under the assumption $v_\circ \in J^2(\mathbb{R}^3)$. Indeed for a suitable weak solution the pointwise asymptotic behavior is of the kind $O(\|v_\circ\|_2 t^{-\frac{3}{4}})$, according to the fact that a weak solution becomes smooth for $t > T_0$, where T_0 is connected with the L^2-norm of $v_\circ$, and the behavior is stated considering the L^2-norm of the initial data.

Introduction to the second result

To better explain our aim, we start from two basic results, one due to Leray (1934) (*structure theorem*) and the other contained in [CKN82].

Leray proves that a weak solution becomes smooth, that is $\|v(t)\|_\infty < \infty$, for all $t > T_0$, where $T_0 \le c\|v_\circ\|_2^4$.

Based on Proposition 1 and Proposition 2, in [CKN82] it is proved that if $\|\nabla v_\circ\|_{L^2(\mathbb{R}^3 - B_R)} < \infty$, then there exists a $R' \ge R$ such that $\|v(t)\|_{L^\infty(\mathbb{R}^3 - B_{R'})} < \infty$ for all $t > 0$.

Further, in the light of our Theorem 2 we can claim:

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Introduction to the second result

To better explain our aim, we start from two basic results, one due to Leray (1934) (*structure theorem*) and the other contained in [CKN82].

Leray proves that a weak solution becomes smooth, that is $\|v(t)\|_\infty < \infty$, for all $t > T_0$, where $T_0 \leq c\|v_\circ\|_2^4$.

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Roughly speaking, following Leray and the results in [CKN82], we say that the possible *turbulence* of a weak solution not only appears in a finite time ($t < T_0$), but also in a bounded region of the space ($B_{R'}$), whose $\mathscr{P}^1$-measure is null (this last is true as soon as a suitable weak solution exists).

Actually, the smallness of the data for large $|x|$, although given by means of integrability conditions outside a ball, preserves the regularity of the weak solution (as in the case of *"small data"*).

The aim of our note goes in the direction of the last claims. We prove that, not only the possible *turbulence* does not perturb the regularity of a weak solution in a neighborhood of *infinity*, but, if an asymptotic spatial behavior of the initial data $v_\circ$ is given, then the same behavior holds for a suitable weak solution for all $t > 0$.

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Theorem 3

Let $v_\circ \in J^2(\mathbb{R}^3)$ *and, for some* $\alpha \in [1,3)$ *and* $R_0 > 0$*, let be* $|v_\circ(x)| \leq V_\circ |x|^{-\alpha}$*, for* $|x| > R_0$*. Let* (v, π_v) *be a suitable weak solution to the Navier-Stokes Cauchy problem. Then, there exists a constant* $M \geq 1$ *such that*

$$|v(t,x)| \leq c(v_\circ)|x|^{-\alpha}, \text{ for all } (t,x) \in (0,\infty) \times \mathbb{R}^3 \setminus B_{MR_0}, \tag{7}$$

where M *is independent of* $v_\circ$*. Moreover,*

$$|v(t,x)| \leq c(v_\circ) t^{-\frac{\alpha}{2}}, (t,x) \in (T_0, \infty) \times \mathbb{R}^3.$$

The second result: On its proof

We work on a given suitable weak solution.

Hence we do not construct new solutions.

We are able to give a representation formula in $\mathbb{R}^3 \setminus B_{M_0 R_0}$, where the existence of M_0 is proved in the first result.

Then, we get for all $t > 0$ and almost a.e. in $x \in \mathbb{R}^3 \setminus B_{M_0 R_0}$:

$$v(t,x) = \mathbb{H}[v_\circ](t,x) + \mathbb{T}[v \otimes v](t,x)\,, \tag{8}$$

$$\mathbb{H}[v_0](t-s,x) := \int_{\mathbb{R}^3} H(t-\tau,y)v_\circ(y)dy =: w(t,x)\,,$$

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For the first term there are no problems on the spatial asymptotic behavior.

For the second term we consider

$$v = (v - w) + w =: u + w.$$

Hence,

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It is not difficult to discuss the terms which contain the linear part w.

As far as it concerns the term $\mathbb{T}[u, u]$, that we call non-smooth term, we employ the following estimate

$$\|u(t)\|^2_{L^2(\mathbb{R}^3 \setminus B_R)} \leq c(v_\circ) R^{-1}$$

This behavior derives from a suitable discussion of the generalized energy inequality (it is not trivial). Of course, an iterative procedure is needed to perform the same order of decay of the initial data:

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Thank for your attention.