Onset of convection for ternary fluid mixtures saturating rotating horizontal porous layers with large pores, under the action of Brinkman law

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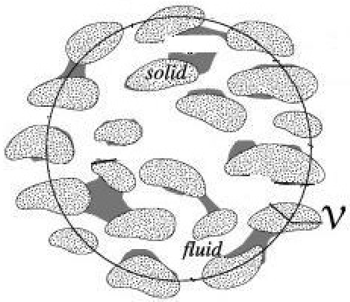
Due to numerous applications in the real-world phenomena, convection-diffusion in fluid mixtures in porous media is a very active area of research.

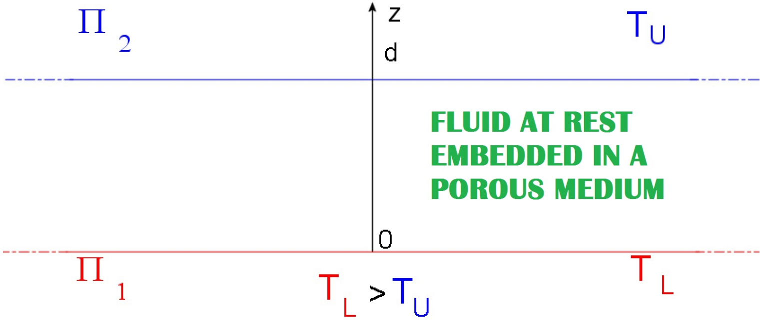
In fact, till now, many papers have been devoted either to the double diffusive-convection or to multi-component diffusive-convection since it appears in numerous physical problems such as the spreading of pollutants, contaminant transport in saturated soil, underground disposal of nuclearwastes and food processing.

Representative Elementary Volume of Porous Medium

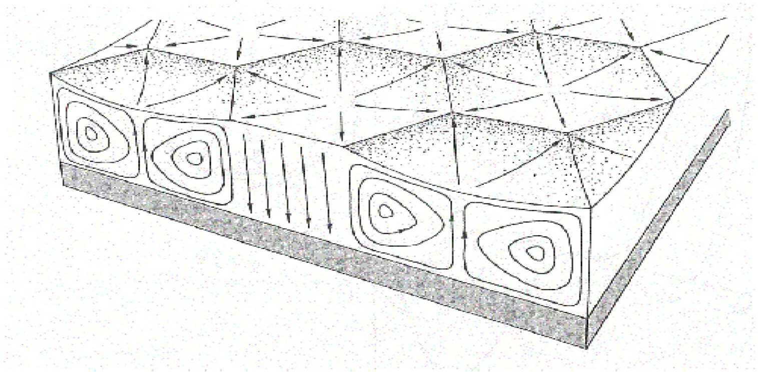


$$\Phi = \frac{\text{Volume of Void Space}}{\text{Total Volume}} \qquad \text{(Porosity)}$$

Horton-Rogers-Lapwood Problem

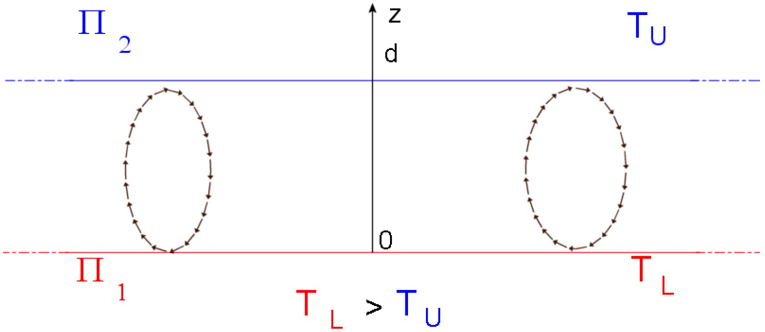


Cells Pattern Formation



....when Convective Motions arise: $R^2 > R_C^2$

$$R^2 = \frac{g\alpha\beta}{k\nu} d^4$$ Rayleigh Number



Lord Rayleigh "On convective currents ... the under side" Phil. Mag **32** (1916)

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Further, since the porous medium is assumed to have large pores, we assume the validity of Brinkman law.

Mathematical Model:

$$\begin{cases} \nabla P = -\dfrac{\mu_1}{K}\mathbf{v} + \mu_2 \Delta \mathbf{v} - 2\rho_0 \omega \mathbf{k} \times \mathbf{v} - \mathbf{g}\rho_f, \\ \nabla \cdot \mathbf{v} = 0, \\ \dfrac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T = K_T \Delta T, \\ \dfrac{\partial C_i}{\partial t} + \mathbf{v} \cdot \nabla C_i = K_i \Delta C_i, \quad i = 1,2 \end{cases} \tag{1}$$

where

$$P = p - \frac{\rho_0}{2}|\underline{\omega} \times \mathbf{x}|^2, \; \underline{\omega} = \omega \mathbf{k} = \text{angular velocity.}$$

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Boundary conditions

$$\begin{cases} T(x,y,0,t) = T_l, \quad T(x,y,d,t) = T_u, \qquad T_l > T_u \\ C_i(x,y,0,t) = C_{il}, \quad C_i(x,y,d,t) = C_{iu}, \; i = 1, 2, \\ \mathbf{v} \cdot \mathbf{k} = 0, \text{ on } z = 0, d \end{cases} \tag{2}$$

Denoting by $(\tilde{v}, \tilde{p}, \tilde{T}, \tilde{C}_i)$ the conduction solution, and setting

$$\mathbf{v} = \tilde{\mathbf{v}} + \mathbf{u},\ \ p = \tilde{p} + \pi,\ \ T = \tilde{T} + \theta,\ \ C_i = \tilde{C}_i + \gamma_i\ \ (i = 1, 2), \qquad (3)$$

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$$\begin{cases} \nabla \pi = -\mathbf{u} + D_a \Delta \mathbf{u} + \tau \mathbf{u} \times \mathbf{k} + (R\theta - R_1\gamma_1 - R_2\gamma_2)\mathbf{k}, \\ \nabla \cdot \mathbf{u} = 0, \\ \theta_t + \mathbf{u} \cdot \nabla \theta = Rw + \Delta \theta, \\ P_i \left(\dfrac{\partial \gamma_i}{\partial t} + \mathbf{u} \cdot \nabla \gamma_i \right) = H_i R_i w + \Delta \gamma_i, \quad i = 1,2, \end{cases} \qquad (4)$$

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where $H_i = \pm 1$, according to the layer is salted from below or above, and

$$D_a = \frac{\mu_2 K}{\mu_1 d^2} \text{(Darcy number)}$$

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$$P_i = \frac{K_T}{K_i} \text{(Prandtl number)}, \ (i = 1, 2)$$

$$R = \left(\frac{\alpha \rho_0 g K d \delta T}{\mu_1 K_T} \right)^{\frac{1}{2}} \text{(thermal Rayleigh number)}$$

$$R_i = \left(\frac{\beta_i \rho_0 g K d P_i \delta C_i}{\mu_1 K_T} \right)^{\frac{1}{2}} \text{(solute Rayleigh numbers)} (i = 1, 2).$$

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i) $\mathbf{u}, \theta, \gamma_1, \gamma_2$ are periodic in the x and y directions of period $\frac{2\pi}{a_x}, \frac{2\pi}{a_y}$ respectively, and

$$\Omega = \left[0, \frac{2\pi}{a_x}\right] \times \left[0, \frac{2\pi}{a_y}\right] \times [0, 1], \tag{5}$$

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will denote the periodicity cell;

ii) $\mathbf{u}, \theta, \gamma_1, \gamma_2$, belong to $W^{2,2}(\Omega)$ and are such that all their first derivatives and second spatial derivatives can be expanded in Fourier series uniformly convergent in Ω.

Main boundary value problem

$$\begin{cases} \nabla\pi = -\mathbf{u} + D_a\Delta\mathbf{u} + \tau\mathbf{u}\times\mathbf{k} + (R\theta - R_1\gamma_1 - R_2\gamma_2)\mathbf{k}, \\ \nabla\cdot\mathbf{u} = 0, \\ w = \theta = \gamma_1 = \gamma_2 = 0, \qquad \text{on } z = 0, 1. \end{cases} \tag{6}$$

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Since the set $\{\sin n\pi z\}_{n\in\mathbb{N}}$ is a complete orthogonal system for $L^2(0,1)$, then

$$\Gamma = \sum_{n=1}^{\infty}\Gamma_n = \sum_{n=1}^{\infty}\tilde{\Gamma}_n(x,y,t)\sin(n\pi z), \quad \forall\,\Gamma\in\{w,\theta,\gamma_1,\gamma_2\}. \tag{7}$$

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By virtue of the periodicity in the x and y directions

$$\tilde{\Gamma}_n(x,y,t) = \Gamma_n^*(t)e^{i(a_x x + a_y y)} \tag{8}$$

Theorem 1

Let $(w,\theta,\gamma_1,\gamma_2)\in[L^*(\Omega)]^4$*, then* $\mathbf{u}=(u,v,w)$*, solution of*

$$\begin{cases}\nabla\pi=-\mathbf{u}+D_a\Delta\mathbf{u}+\tau\mathbf{u}\times\mathbf{k}+(R\theta-R_1\gamma_1-R_2\gamma_2)\mathbf{k},\\ \nabla\cdot\mathbf{u}=0,\\ w=\theta=\gamma_1=\gamma_2=0, \qquad on\ z=0,1\,,\end{cases}\tag{9}$$

is given by

$$\begin{cases}u=\sum_{n=1}^{\infty}u_n(x,y,z,t), \quad v=\sum_{n=1}^{\infty}v_n(x,y,z,t)\\ w=\sum_{n=1}^{\infty}w_n(x,y,z,t)=\sum_{n=1}^{\infty}\tilde{w}_n(x,y,t)\sin(n\pi z),\end{cases}\tag{10}$$

where

Theorem 1

$$\begin{cases} u_n = \dfrac{1}{a^2}\dfrac{\partial^2 w_n}{\partial x \partial z} + \dfrac{\tau}{a^2(1+D_a\xi_n)}\dfrac{\partial^2 w_n}{\partial y \partial z} \\ v_n = \dfrac{1}{a^2}\dfrac{\partial^2 w_n}{\partial y \partial z} - \dfrac{\tau}{a^2(1+D_a\xi_n)}\dfrac{\partial^2 w_n}{\partial x \partial z} \\ w_n = \eta_n(R\theta_n - R_1\gamma_{1n} - R_2\gamma_{2n}) \end{cases} \tag{11}$$

$$a^2 = a_x^2 + a_y^2$$

$$\xi_n = a^2 + n^2\pi^2$$

$$\eta_n = \frac{a^2(1+D_a\xi_n)}{\xi_n(1+D_a\xi_n)^2 + n^2\pi^2\tau^2}.$$

Remark

In view of Theorem 2, it follows that the independent fields of (4) are reduced to the three fields $\theta, \gamma_1, \gamma_2$.

Setting

$$\begin{cases} a_{1n} = R^2\eta_n - \xi_n, \ a_{2n} = -RR_1\eta_n, \ a_{3n} = -RR_2\eta_n \\ b_{1n} = \dfrac{H_1RR_1\eta_n}{P_1}, b_{2n} = \dfrac{-(H_1R_1^2\eta_n + \xi_n)}{P_1}, b_{3n} = \dfrac{-H_1R_1R_2\eta_n}{P_1} \\ c_{1n} = \dfrac{H_2RR_2\eta_n}{P_2}, c_{2n} = \dfrac{-H_2R_1R_2\eta_n}{P_2}, c_{3n} = \dfrac{-(H_2R_2^2\eta_n + \xi_n)}{P_2} \end{cases}$$

and

$$\mathbb{L}_n = \begin{pmatrix} a_{1n} & a_{2n} & a_{3n} \\ b_{1n} & b_{2n} & b_{3n} \\ c_{1n} & c_{2n} & c_{3n} \end{pmatrix},$$

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the evolution system can be written as follows

$$\frac{\partial}{\partial t}\begin{pmatrix}\theta\\ \gamma_1\\ \gamma_2\end{pmatrix} = \sum_{n=1}^{\infty} \mathbb{L}_n \begin{pmatrix}\theta_n\\ \gamma_{1n}\\ \gamma_{2n}\end{pmatrix} - \begin{pmatrix}\mathbf{u}\cdot\nabla\theta\\ \mathbf{u}\cdot\nabla\gamma_1\\ \mathbf{u}\cdot\nabla\gamma_2\end{pmatrix}, \tag{12}$$

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to which we add the initial-boundary conditions

$$\begin{cases}\theta_0 = \sum_{n=1}^{\infty}(\theta_n)_{t=0} = \sum_{n=1}^{\infty}\theta_{0n},\\ \gamma_{i0} = \sum_{n=1}^{\infty}(\gamma_{in})_{(t=0)} = \sum_{n=1}^{\infty}\gamma_{i0n}, \ i = 1, 2,\\ \theta = \gamma_i = 0, \ \text{on } z = 0, 1,\end{cases} \qquad (13)$$

$$\frac{\partial}{\partial t}\begin{pmatrix}\theta\\ \gamma_1\\ \gamma_2\end{pmatrix} = \sum_{n=1}^{\infty} \mathbb{L}_n \begin{pmatrix}\theta_n\\ \gamma_{1n}\\ \gamma_{2n}\end{pmatrix} - \begin{pmatrix}\mathbf{u}\cdot\nabla\theta\\ \mathbf{u}\cdot\nabla\gamma_1\\ \mathbf{u}\cdot\nabla\gamma_2\end{pmatrix}, \tag{12}$$

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where $\mathbf{u} = (u, v, w)$ is the divergence free vector determined by solving

$$\begin{cases}(D_a\Delta - 1)^2\Delta w + \tau^2 w_{zz} + (D_a\Delta - 1)\Delta_1(R\theta - R_1\gamma_1 - R_2\gamma_2) = 0\\ w = \theta = \gamma_1 = \gamma_2 = 0, \qquad \text{on } z = 0, 1.\end{cases} \tag{14}$$

Uniqueness Theorem

The i.b.v. problem (12)-(13) admits a unique solution $(\theta, \gamma_1, \gamma_2) \in [L^*(\Omega)]^3$.

On accounting for the "Auxiliary System Method" introduced by Rionero S.:

- Rionero, S.: J. Eng. Sc. 48 (2010), 1519-1533.
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- Rionero, S.: Phys. Fluids 24 (2012), 104101.
- Rionero, S.: Phys. Fluids 25 (2013), 054104.
- Rionero, S.: Ricerche di Matematica 62 (2013), 183-208.
- Rionero, S.: Rend. Lincei Mat. Appl. 25 (2014), 1-44.

To (12), we associate, $\forall n \in \mathbb{N}$, the following "auxiliary system", i.e. auxiliary evolution system of the n-th Fourier component of the perturbation fields:

$$\frac{\partial}{\partial t}\begin{pmatrix}\theta_n\\ \gamma_{1n}\\ \gamma_{2n}\end{pmatrix} = \mathbb{L}_n\begin{pmatrix}\theta_n\\ \gamma_{1n}\\ \gamma_{2n}\end{pmatrix} - \begin{pmatrix}\mathbf{u}\cdot\nabla\theta_n\\ \mathbf{u}\cdot\nabla\gamma_{1n}\\ \mathbf{u}\cdot\nabla\gamma_{2n}\end{pmatrix}, \qquad (15)$$

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under the initial-boundary conditions

$$\begin{cases}(\theta_n)_{t=0} = \theta_{0n}, \quad (\gamma_{in})_{t=0} = \gamma_{0in},\ i = 1,2,\\ \theta_n = \gamma_{in} = 0,\ (i = 1,2),\ \text{ on } z = 0,1.\end{cases} \tag{16}$$

Theorem 2

Let $(\theta_n, \gamma_{1n}, \gamma_{2n})$ *be,* $\forall n \in \mathbb{N}$*, solution of (15)-(16). Then the series* $\sum_{n=1}^{\infty} \theta_n$, $\sum_{n=1}^{\infty} \gamma_{1n}$ *and* $\sum_{n=1}^{\infty} \gamma_{2n}$ *are convergent and it follows that*

$$\sum_{n=1}^{\infty} \theta_n = \theta, \ \sum_{n=1}^{\infty} \gamma_{in} = \gamma_i, \ i = 1, 2, \tag{17}$$

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with $(\theta, \gamma_1, \gamma_2)$ *solution of (12)-(13).*

Remark

The global nonlinear stability of the conduction solution is guaranteed if exist conditions - independent of n - guaranteeing the global nonlinear stability of the null solution of the "Auxiliary System".

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Setting

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$$\begin{cases} X_n = \theta_n, \quad Y_n = \sqrt{P_1}\gamma_{1n}, \quad Z_n = \sqrt{P_2}\gamma_{2n}, \\ X = \sum_{n=1}^{\infty} X_n, \quad Y = \sum_{n=1}^{\infty} Y_n, \quad Z = \sum_{n=1}^{\infty} Z_n, \end{cases}$$

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system (12) becomes

$$\frac{\partial}{\partial t}\begin{pmatrix} X_n \\ Y_n \\ Z_n \end{pmatrix} = A_n \begin{pmatrix} X_n \\ Y_n \\ Z_n \end{pmatrix} - \begin{pmatrix} \mathbf{u}\cdot X_n \\ \mathbf{u}\cdot Y_n \\ \mathbf{u}\cdot Z_n \end{pmatrix}, \tag{18}$$

with the linear operator A_n given by the symmetric matrix with the linear operator A_n given by the symmetric matrix

$$A_n = \begin{pmatrix} R^2\eta_n - \xi_n & -\frac{RR_1}{\sqrt{P_1}}\eta_n & -\frac{RR_2}{\sqrt{P_2}}\eta_n \\ -\frac{RR_1}{\sqrt{P_1}}\eta_n & \frac{R_1^2\eta_n - \xi_n}{P_1} & \frac{R_1R_2}{\sqrt{P_1P_2}}\eta_n \\ -\frac{RR_2}{\sqrt{P_2}}\eta_n & \frac{R_1R_2}{\sqrt{P_1P_2}}\eta_n & \frac{R_2^2\eta_n - \xi_n}{P_2} \end{pmatrix}.$$

On setting $A_n = \begin{pmatrix} \alpha_{11n} & \alpha_{12n} & \alpha_{13n} \\ \alpha_{12n} & \alpha_{22n} & \alpha_{23n} \\ \alpha_{13n} & \alpha_{23n} & \alpha_{33n} \end{pmatrix}$, and denoting by

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$$\mathrm{I}_{1n} < 0, \ \mathrm{I}_{3n} < 0, \ \mathrm{I}_{1n}\mathrm{I}_{2n} - \mathrm{I}_{3n} < 0, \ \forall n \in \mathbb{N}, \tag{20}$$

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which imply $\mathrm{I}_{2n} > 0, \forall n \in \mathbb{N}$.

Setting

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The conduction solution is globally, nonlinearly, asymptotically $L^2(\Omega)-$stable if and only if

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Stabilizing effects of rotation and Brinkman terms

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$$\mathscr{A}(n^2, a^2, D_a, \tau) = \frac{\xi_n^2(1 + D_a\xi_n)}{a^2} + \frac{n^2\pi^2\mathscr{T}^2\xi_n}{a^2(1 + D_a\xi_n)}, \tag{24}$$

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$$A^* > \min \mathscr{A}(n^2, a^2, 0, 0) = 4\pi^2. \tag{26}$$

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- in view of

$$\frac{\partial \mathscr{A}_1}{\partial D_a} = \frac{(a^2+\pi^2)^2}{a^2}\left(a^2+\pi^2 - \frac{\pi^2\tau^2}{\left[1+D_a(a^2+\pi^2)\right]^2}\right),$$

it follows that, if

$$D_a > D_a^* = \frac{\tau-1}{\pi^2},$$

then $\mathscr{A}_1$ is an increasing function of D_a.

Theorem 4

In the absence of Brinkman law, the global stability of the conduction solution is guaranteed if and only if

$$R^2 + R_1^2 + R_2^2 < A_\tau^* = \pi^2(1 + \sqrt{1 + \tau^2})^2. \quad (27)$$

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τ	D_a	A_τ^*	$R_C^2 = R^2 + R_1^2 + R_2^2 < A_\tau^*$
0	0	$4\pi^2$	$R_C^2 < 4\pi^2$
0.1	0	39.6756	$R_C^2 < 39.6756$
0.2	0	40.2641	$R_C^2 < 40.2641$
0.5	0	44.2757	$R_C^2 < 44.2757$
1.2	0	64.7851	$R_C^2 < 64.7851$
1.5	0	77.5312	$R_C^2 < 77.5312$

Table: Stability condition (27).

Theorem 5

In the absence of rotation, the global stability of the conduction solution is guaranteed if and only if

$$R^2 + R_1^2 + R_2^2 < A_{D_a}^* = \frac{(X^*)^2(1 + D_a X^*)}{X^* - \pi^2}, \qquad (28)$$

with

$$X^* = \frac{3D_a\pi^2 - 1 + \sqrt{(3D_a\pi^2 - 1)^2 + 16\pi^2 D_a}}{4D_a}. \qquad (29)$$

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τ	D_a	$A^*_{D_a}$	$R_c^2 = R^2 + R_1^2 + R_2^2 < A^*_{D_a}$
0	0	$4\pi^2$	$R_c^2 < 4\pi^2$
0	0.1	108.573	$R_c^2 < 108.573$
0	0.5	372.722	$R_c^2 < 372.722$
0	1.5	1030.52	$R_c^2 < 1030.52$

Theorem 6

Let $\tau \leq 1$. Then

$$R^2 + R_1^2 + R_2^2 < \frac{A_\tau^* + A_{D_a}^*}{2}, \tag{30}$$

is sufficient for guaranteeing the global stability of the conduction solution in the presence of rotation and Brinkman law.

Theorem 6

Let $\tau \leq 1$. *Then*

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is sufficient for guaranteeing the global stability of the conduction solution in the presence of rotation and Brinkman law.

τ	D_a	$\frac{A_\tau^* + A_{D_a}^*}{2}$	$R_C^2 = R^2 + R_1^2 + R_2^2 < \frac{A_\tau^* + A_{D_a}^*}{2}$
0	0	$4\pi^2$	$R_C^2 < 4\pi^2 = 39.4384$
0.1	0.1	74.1243	$R_C^2 < 74.1243$
0.2	0.5	206.493	$R_C^2 < 206.493$
0.3	1	371.462	$R_C^2 < 371.462$

Theorem 7

Let

$$1 < \tau < 1 + D_a \pi^2. \quad (31)$$

On setting $D_a^* = \dfrac{\tau - 1}{\pi^2}$ *and* $A^*_{D_a^*} = \dfrac{(Y^*)^2 \left[1 + D_a^* Y^*\right]}{Y^* - \pi^2}$, *then either*

$$R^2 + R_1^2 + R_2^2 < \max\left\{A^*_\tau; A^*_{D_a^*}\right\} \quad (32)$$

or

$$R^2 + R_1^2 + R_2^2 < \frac{A^*_\tau + A^*_{D_a^*}}{2}, \quad (33)$$

with Y^* *given by (29) where* $D_a = D_a^*$, *guarantees the global stability of the conduction solution.*

τ	D_a	D_a^*	$\frac{A_\tau^* + A_{D_a^*}^*}{2}$	$R_c^2 < \frac{A_\tau^* + A_{D_a^*}^*}{2}$
1.1	0.02	0.0101321	54.0569	$R_c^2 < 54.0569$
1.2	0.1	0.0202642	59.5725	$R_c^2 < 59.5725$
1.5	0.5	0.0506606	76.4521	$R_c^2 < 76.4521$
2	3	0.101321	106.406	$R_c^2 < 106.406$
2.1	3.5	0.111453	112.695	$R_c^2 < 112.695$
2.5	4	0.151982	138.863	$R_c^2 < 138.863$
3	5	0.202642	173.841	$R_c^2 < 173.841$
3.1	10	0.212774	181.138	$R_c^2 < 181.138$

Table: Stability condition (33)

Layer salted from above and below.

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$$
\mathbb{L}_n = \begin{pmatrix} R^2\eta_n - \xi_n & -RR_1\eta_n & -RR_2\eta_n \\ \dfrac{RR_1\eta_n}{P_1} & -\dfrac{R_1^2\eta_n + \xi_n}{P_1} & -\dfrac{R_1R_2\eta_n}{P_1} \\ -\dfrac{RR_2\eta_n}{P_2} & \dfrac{R_1R_2\eta_n}{P_2} & \dfrac{R_2^2\eta_n - \xi_n}{P_2} \end{pmatrix}.
$$

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$$\theta_n^* = \theta_n, \quad \Phi_{in}^* = \frac{1}{\mu_{in}}\Phi_{in}, \quad (i = 1,2),$$

with

$$\mu_{1n} = \sqrt{|1 - P_1|\frac{\xi_n}{\eta_n}}, \qquad \mu_{2n} = \sqrt{|P_2 - 1|\frac{\xi_n}{\eta_n}},$$

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the "Auxliary System", omitting the stars, becomes

$$\frac{\partial}{\partial t}\begin{pmatrix}\theta_n\\ \Phi_{1n}\\ \Phi_{2n}\end{pmatrix} = \tilde{\mathscr{L}}_n\begin{pmatrix}\theta_n\\ \Phi_{1n}\\ \Phi_{2n}\end{pmatrix} - \begin{pmatrix}\mathbf{u}\cdot\nabla\theta_n\\ \mathbf{u}\cdot\nabla\Phi_{1n}\\ \mathbf{u}\cdot\nabla\Phi_{2n}\end{pmatrix}. \qquad (34)$$

where $\tilde{\mathscr{L}}_n$ is given by

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$$\tilde{\mathscr{L}}_n=\begin{pmatrix} R^*\eta_n-\xi_n & \frac{R_1}{P_1}\sqrt{|1-P_1|\xi_n\eta_n} & -\frac{R_2}{P_2}\sqrt{|P_2-1|\xi_n\eta_n} \\ \frac{R_1(1-P_1)\sqrt{\xi_n\eta_n}}{P_1\sqrt{|1-P_1|}} & -\frac{\xi_n}{P_1} & 0 \\ -\frac{R_2(P_2-1)\sqrt{\xi_n\eta_n}}{P_2\sqrt{|P_2-1|}} & 0 & -\frac{\xi_n}{P_2} \end{pmatrix}.$$

where $\tilde{\mathscr{L}}_n$ is given by

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and

$$R^* = R^2 - \frac{R_1^2}{P_1} + \frac{R_2^2}{P_2}. \tag{35}$$

Setting

$$A^* = \inf_{(a^2,n)\in\mathbb{R}^+\times\mathbb{N}} \frac{\xi_n}{\eta_n} \tag{36}$$

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$$E = \sum_{n=1}^{\infty} E_n, \tag{37}$$

with

$$E_n = \int_\Omega (\theta_n^2 + \Phi_{1n}^2 + \Phi_{2n}^2)\, d\Omega, \tag{38}$$

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the following theorem holds true.

Theorem 8

The global nonlinear stability of the conduction solution is guaranteed by

$$R^2 < R_1^2 - R_2^2 + A^*, \qquad for \qquad P_1 \leq 1, P_2 \geq 1, \tag{39}$$

$$R^2 < \frac{R_1^2}{P_1} - \frac{R_2^2}{P_2} + A^*, \qquad for \qquad P_1 \geq 1, P_2 \leq 1, \tag{40}$$

$$R^2 < R_1^2 - \frac{R_2^2}{P_2} + A^*, \qquad for \qquad P_1 \leq 1, P_2 \leq 1, \tag{41}$$

$$R^2 < \frac{R_1^2}{P_1} - R_2^2 + A^*, \qquad for \qquad P_1 \geq 1, P_2 \geq 1, \tag{42}$$

(39) being also necessary.

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