$\mathfrak{sl}_N$ web algebras and categorified Howe duality

Marco Mackaay
(joint with Pan-Tubbenhauer and Yonezawa)

CAMGSD and University of the Algarve, Portugal

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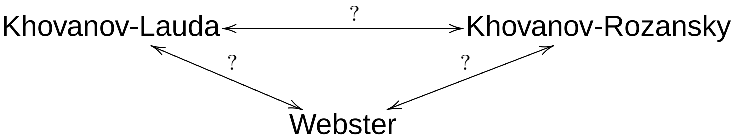
Motivation

Main question

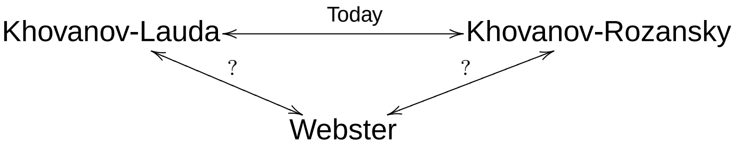
What is the relation between the three combinatorial approaches to categorification in type A?



Motivation

Part of the answer

Categorified skew Howe duality!



The general and special linear quantum groups

Definition

i) $\mathbf{U}_q(\mathfrak{gl}_n)$ is generated by $K_1^{\pm 1}, \dots, K_n^{\pm 1}, E_{\pm 1}, \dots, E_{\pm(n-1)}$, subject to ($\alpha_i = \epsilon_i - \epsilon_{i+1} = (0, \dots, 1, -1, \dots, 0) \in \mathbb{Z}^{n-1}$):

$$K_iK_j = K_jK_i \quad K_iK_i^{-1} = K_i^{-1}K_i = 1$$

$$E_iE_{-j} - E_{-j}E_i = \delta_{i,j}\frac{K_iK_{i+1}^{-1} - K_i^{-1}K_{i+1}}{q - q^{-1}}$$

$$K_iE_{\pm j} = q^{\pm(\epsilon_i, \alpha_j)}E_{\pm j}K_i$$

$+$ some extra relations we won't need today

ii) $\mathbf{U}_q(\mathfrak{sl}_n) \subseteq \mathbf{U}_q(\mathfrak{gl}_n)$ is generated by $K_iK_{i+1}^{-1}$ and $E_{\pm i}$.

Idempotented quantum groups

Definition (Beilinson-Lusztig-MacPherson)

For each $\lambda \in \mathbb{Z}^n$, adjoin an idempotent 1_λ and add the relations

$$1_\lambda 1_\mu = \delta_{\lambda,\nu} 1_\lambda$$
$$E_{\pm i} 1_\lambda = 1_{\lambda \pm \alpha_i} E_{\pm i}$$
$$K_i 1_\lambda = q^{\lambda_i} 1_\lambda.$$

Define

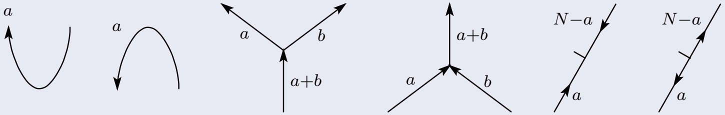
$$\dot{\mathbf{U}}_q(\mathfrak{gl}_n) = \bigoplus_{\lambda,\mu \in \mathbb{Z}^n} 1_\lambda \mathbf{U}_q(\mathfrak{gl}_n) 1_\mu.$$

Define $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$ similarly by adjoining idempotents 1_μ to $\mathbf{U}_q(\mathfrak{sl}_n)$ for $\mu \in \mathbb{Z}^{n-1}$.

Definition (Cautis-Kamnitzer-Morrison)

The objects of $\mathcal{S}p(\mathrm{SL}_N)$ are finite sequences $\vec{k}$ of elements in $\{0^{\pm}, \ldots, (N)^{\pm}\}$.

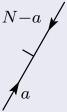
$\mathrm{Hom}(\vec{k}, \vec{l})$ is generated by:



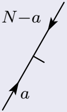
with all labels between 0 and N.

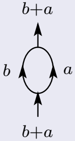
Definition

Modulo:

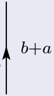


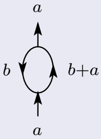
$$= (-1)^{a(N-a)}$$





$$= \begin{bmatrix} a+b \\ a \end{bmatrix}_q$$





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$+$ rels by reflections and/or arrow reversals;

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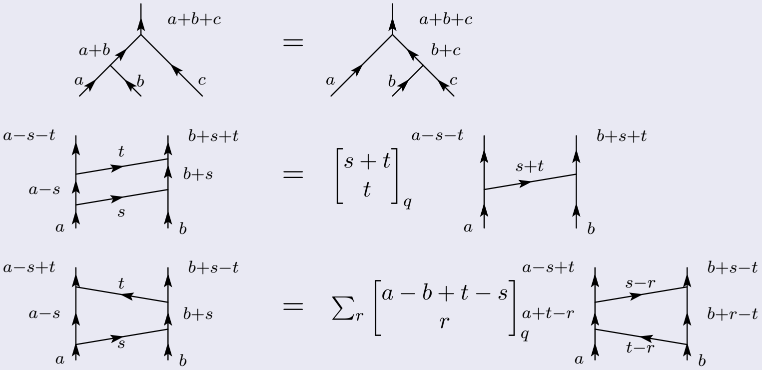


$+$ rels by reflections and/or arrow reversals.

let $N, m, d \geq 0$ be arbitrary integers.

Definition (N-bounded $\mathfrak{gl}_m$-weights)

Let

$$\Lambda(m,d)_N := \{\vec{k} \in \{0,\dots,N\}^m \mid k_1 + \cdots + k_m = d\}.$$

Define

$$\phi_{m,d,N} \colon \mathbb{Z}^{m-1} \to \Lambda(m,d)_N \cup \{*\}$$

by

$$\phi_{m,d,N}(\lambda) = \vec{k} \qquad \text{if}$$

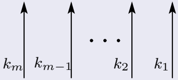
$$k_i - k_{i+1} = \lambda_i \quad \text{and} \quad \sum_{i=1}^{m} k_i = d.$$

Proposition (Cautis-Kamnitzer-Morrison)

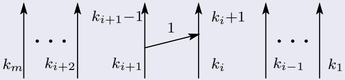
$$\gamma_{m,d,N} \colon \dot{\mathrm{U}}_q(\mathfrak{sl}_m) \to \mathcal{S}p(\mathrm{SL}_N)$$

is a well-defined full functor: (assume $\phi_{m,d,N}(\lambda) = \vec{k}$)

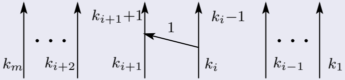
$1_\lambda \quad \mapsto$



$E_{+i}1_\lambda \quad \mapsto$



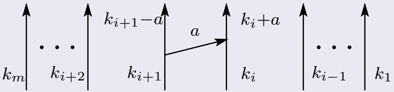
$E_{-i}1_\lambda \quad \mapsto$



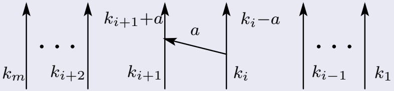
Recall

$$E_{+i}^{(a)} := E_{+i}^{a}/[a]! \quad \text{and} \quad E_{-i}^{(a)} = E_{-i}^{a}/[a]!$$

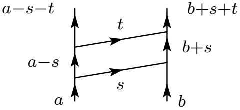
$E_{+i}^{(a)} 1_\lambda \quad \mapsto$



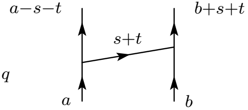
$E_{-i}^{(a)} 1_\lambda \quad \mapsto$



Examples



$$= \begin{bmatrix} s+t \\ t \end{bmatrix}_q$$



$$E_+^{(s)} E_+^{(t)} 1_{(b,a)} = \begin{bmatrix} s+t \\ t \end{bmatrix}_q E_+^{(s+t)}$$

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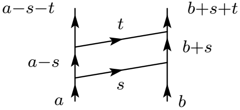

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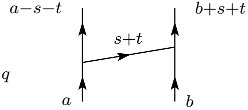

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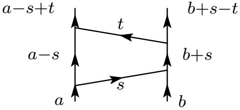
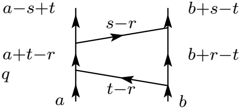


$$= \sum_r \begin{bmatrix} a-b+t-s \\ r \end{bmatrix}_q$$



$$E_+^{(s)} E_-^{(t)} 1_{(b,a)} = \sum_r \begin{bmatrix} a-b+t-s \\ r \end{bmatrix}_q E_-^{(t-r)} E_+^{(s-r)}$$

A special case

Suppose $d = m = N\ell$ and $\Lambda = N\omega_\ell$. Note that

$$\phi_{m,m,N}(\Lambda) = (N^\ell) \in \Lambda(m,m)_N.$$

Definition (M.M-Yonezawa)

Define the $\dot{\mathrm{U}}_q(\mathfrak{sl}_m)$-*web module* with highest weight Λ by

$$W_\Lambda := \bigoplus_{\vec{k}\in\Lambda(m,m)_N} W(\vec{k},N),$$

where $W(\vec{k},N)$ is the *web space* defined by

$$W(\vec{k},N) := \mathrm{Hom}((N^\ell),\vec{k}).$$

in $\mathcal{S}p(\mathrm{SL}(N))$.

The web module is irreducible

Definition (M.M.-Yonezawa)

Define the q-*sesquilinear web form* by

$$\langle u, v\rangle := q^{d(\vec{k})}\mathrm{ev}(u^*v) \in \mathbb{C}(q) \tag{1}$$

for any two monomial webs $u, v \in W(\vec{k}, N)$, with

$$d(\vec{k}) = 1/2(N(N-1)\ell - \sum_{i=1}^{m} k_i(k_i - 1)).$$

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Corollary (M.M-Yonezawa)

W_Λ is an irreducible $\dot{\mathrm{U}}_q(\mathfrak{sl}_m)$-representation with highest weight Λ. The q-sesquilinear web form is equal to the q-Schapovalov form.

And now the categorification...

Matrix factorizations

Let $R = \mathcal{C}[\mathbb{X}] = \mathbb{C}[X_1, ..., X_k]$ and suppose $\deg(X_i) \in 2\mathbb{N}$.

Definition

A *graded matrix factorization with homogeneous potential* $P \in R$ is a 2-chain of free graded R-modules

$$M_0 \xrightarrow{d_{M_0}} M_1 \xrightarrow{d_{M_1}} M_0 \ ,$$

with

$$\deg(d_{M_0}) = \deg(d_{M_1}) = \frac{1}{2}\deg(P)$$

and

$$d_{M_1} d_{M_0} = P\,\mathrm{Id}_{M_0} \quad \text{and} \quad d_{M_0} d_{M_1} = P\,\mathrm{Id}_{M_1}.$$

The homotopy category of matrix factorizations with potential P, denoted $\mathrm{HMF}_R(P)$, is *Krull-Schmidt*.

Matrix factorizations

Let $R = \mathcal{C}[\mathbb{X}]$, $R' = \mathcal{C}[\mathbb{Y}]$, $Q = \mathcal{C}[\mathbb{X} \cup \mathbb{Y}]$ and $S = \mathcal{C}[\mathbb{X} \cap \mathbb{Y}]$.

Definition

For $\widehat{M} \in \mathrm{HMF}_R(P)$ and $\widehat{N} \in \mathrm{HMF}_{R'}(P')$, we define

$$\widehat{M} \underset{S}{\boxtimes} \widehat{N} \in \mathrm{HMF}_Q(P + P').$$

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Definition

Given $\widehat{N}=(N_0,N_1,d_{N_0},d_{N_1})\in \mathrm{HMF}_R(P)$, one can define its dual by

$$\widehat{N}_{\bullet}=(N_0^*,N_1^*,-d_{N_1}^*,d_{N_0}^*)\in \mathrm{HMF}_R(-P),$$

where $N^*=\mathrm{HOM}_R(N,R)$.

We define the structure of a 2-complex on $\mathrm{HOM}_R(\widehat{M}, \widehat{N})$ by

Definition

$$\mathrm{HOM}^0_R(\widehat{M}, \widehat{N}) \xrightarrow{d_0} \mathrm{HOM}^1_R(\widehat{M}, \widehat{N}) \xrightarrow{d_1} \mathrm{HOM}^0_R(\widehat{M}, \widehat{N}),$$

where

$$\mathrm{HOM}^0_R(\widehat{M}, \widehat{N}) = \mathrm{HOM}_R(M_0, N_0) \oplus \mathrm{HOM}_R(M_1, N_1),$$
$$\mathrm{HOM}^1_R(\widehat{M}, \widehat{N}) = \mathrm{HOM}_R(M_0, N_1) \oplus \mathrm{HOM}_R(M_0, N_1),$$

and

$$d_i(f) = d_N\, f + (-1)^i f\, d_M \quad (i = 0, 1).$$

The cohomology of this complex is denoted by

$$\mathrm{EXT}(\widehat{M}, \widehat{N}) = \mathrm{EXT}^0(\widehat{M}, \widehat{N}) \oplus \mathrm{EXT}^1(\widehat{M}, \widehat{N}).$$

By definition, we have the following proposition.

Proposition

We have

$$\begin{aligned} \mathrm{EXT}^0(\widehat{M}, \widehat{N}) &\simeq \mathrm{HOM}_{\mathrm{HMF}}(\widehat{M}, \widehat{N}), \\ \mathrm{EXT}^1(\widehat{M}, \widehat{N}) &\simeq \mathrm{HOM}_{\mathrm{HMF}}(\widehat{M}, \widehat{N}\langle 1 \rangle). \end{aligned}$$

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Lemma

If M is finite, we have an isomorphism

$$\mathrm{EXT}(\widehat{M},\widehat{N}) \cong H(\widehat{M}_\bullet \underset{R}{\boxtimes} \widehat{N})$$

which preserves the q-degree.

Examples

Koszul matrix factorizations

Let $p, q \in R$ be homogeneous. We define

$$K(p;q) := \; R \xrightarrow{\;p\;} R' \xrightarrow{\;q\;} R$$

with $R' := R\{\frac{1}{2}(\,\deg(q) - \deg(p)\,)\}$.

Koszul matrix factorizations

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with $R' := R\{\frac{1}{2}(\deg(q) - \deg(p))\}$.
More generally, for $\mathbf{p} = (p_1, p_2, ..., p_r), \mathbf{q} = (q_1, q_2, ..., q_r) \in R^n$ we define

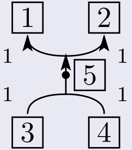
$$K(\mathbf{p};\mathbf{q}) \quad := \quad \underset{R}{\boxtimes}{}_{i=1}^{r} K(p_i; q_i)_R.$$

To each monomial web $u \in \mathcal{S}p(\vec{k}, \vec{k}')$ we can associate a matrix factorization $\hat{u}$.

Example (Khovanov-Rozansky)

Define $f(x+y, xy) := x^{N+1} + y^{N+1}$. Then

$$\begin{aligned}
\widehat{\Gamma} &:= K((p_1, p_2); (x_1 + x_2 - x_3 - x_4, x_1x_2 - x_3x_4))\{-1\} \\
p_1 &:= \frac{f(x_1 + x_2, x_1x_2) - f(x_3 + x_4, x_1x_2)}{x_1 + x_2 - x_3 - x_4} \\
p_2 &:= \frac{f(x_3 + x_4, x_1x_2) - f(x_3 + x_4, x_3x_4)}{x_1x_2 - x_3x_4}
\end{aligned}$$

$\Gamma :=$ 

MFs due to Khovanov-Rozansky, Wu, Yonezawa

We have

$$\mathrm{EXT}(\hat{u},\hat{v}) \cong H(\hat{u}_\bullet \boxtimes_{R^{\vec{k}}} \hat{v}) \cong H(\widehat{u^* v})\{d(\vec{k})\}\langle 1\rangle$$

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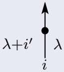
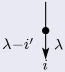
Theorem (Khovanov-Rozansky, Wu, Yonezawa)

The matrix factorizations associated to webs satisfy all web relations up to homotopy equivalence. These equivalences are all q-degree preserving, but might involve homological degree shifts.

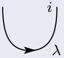
Definition (Khovanov-Lauda's $\mathcal{U}(\mathfrak{sl}_m)$)



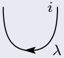
$: \mathcal{E}_{+i}\mathbf{1}_\lambda \to \mathcal{E}_{+i}\mathbf{1}_\lambda\{a_{ii}\}$



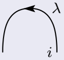
$: \mathcal{E}_{+i}\mathbf{1}_\lambda \to \mathcal{E}_{-i}\mathbf{1}_\lambda\{a_{ii}\}$



$: \mathbf{1}_\lambda \to \mathcal{E}_{(-i,+i)}\mathbf{1}_\lambda\{\lambda_i+1\}$



$: \mathbf{1}_\lambda \to \mathcal{E}_{(+i,-i)}\mathbf{1}_\lambda\{-\lambda_i+1\}$



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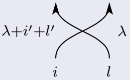
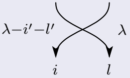


$: \mathcal{E}_{(+i,+l)}\mathbf{1}_\lambda \to \mathcal{E}_{(+l,+i)}\mathbf{1}_\lambda\{-a_{il}\}$



$: \mathcal{E}_{(-i,-l)}\mathbf{1}_\lambda \to \mathcal{E}_{(-l,-i)}\mathbf{1}_\lambda\{-a_{il}\}$

modulo relations.

Theorem (Khovanov-Lauda)

The linear map

$$\dot{\mathbf{U}}_q(\mathfrak{sl}_m) \to K_0^q(\mathcal{U}(\mathfrak{sl}_m))$$

defined by

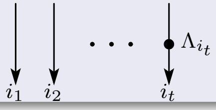
$$q^t E_{\underline{i}} 1_\lambda \to \mathcal{E}_{\underline{i}} \mathbf{1}_\lambda\{t\}$$

is an isomorphism of algebras.

Let Λ be a dominant $\mathfrak{sl}_m$-weight and P_Λ the set of weights in V_Λ.

Definition (The cyclotomic Khovanov-Lauda-Rouquier algebra)

R_Λ is the quotient of $\mathcal{U}_Q(\mathfrak{sl}_m)^-$ modulo the ideal generated by



Note that

$$R_\Lambda = \bigoplus_{\mu \in P_\Lambda} R_\Lambda(\mu),$$

where $R_\Lambda(\mu)$ is the subalgebra generated by all diagrams whose left-most region is labeled μ.

The categorification theorem

Brundan and Kleshchev proved that R_Λ is finite-dimensional.

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Define

$$\mathcal{V}_\Lambda^p := R_\Lambda - \mathrm{pmod}_{\mathrm{gr}}.$$

Theorem (Brundan-Kleshchev)

The 2-category $\mathcal{V}_\Lambda^p$ is an additive strong 2-representation of $\mathfrak{sl}_m$, such that

$$V_\Lambda \cong K_0^q(\mathcal{V}_\Lambda^p)$$

as $\dot{\mathrm{U}}_q(\mathfrak{sl}_m)$-modules.
Moreover, this isomorphism maps intertwines the q-Shapovalov form and the Euler form.

Rouquier's universality theorem

Suppose $\mathcal{C}_\Lambda := \bigoplus_{\mu \in P_\Lambda} C(\mu)$ is a graded and Krull-Schmidt additive category.

Proposition (Rouquier)

Suppose that

- $\mathcal{C}_\Lambda$ *is a strong* 2*-representation of* $\mathfrak{sl}_m$ *by* $\mathcal{C}$*-linear functors;*
- *There exists an indecomposable object* $V(\Lambda) \in \mathcal{C}(\Lambda)$ *such that* $\mathcal{E}_{+i} V(\Lambda) = 0$*, for all* i*, and* $\mathrm{End}(V(\Lambda)) \cong \mathbb{C}$*;*
- *any object in* $\mathcal{C}_\Lambda$ *is a direct summand of* $XV(\Lambda)$*, for some* 1*-morphism* $X \in \mathcal{U}_Q(\mathfrak{sl}_m)$*.*

Then there exists an equivalence

$$\mathcal{V}_\Lambda^p \to \mathcal{C}_\Lambda$$

of additive strong $\mathfrak{sl}_m$ 2*-representations.*

Main theorem

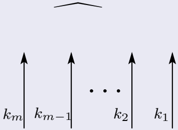
Theorem (M.M.-Yonezawa)

There exists a well-defined linear 2-functor

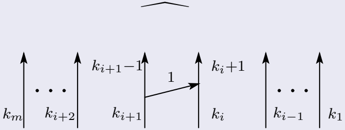
$$\Gamma_{m,d,N} \colon \mathcal{U}(\mathfrak{sl}_m)^* \to \mathrm{HMF}^*_{m,d,N}.$$

Definition

$1_\lambda \quad \mapsto$



$E_{+i}1_\lambda \quad \mapsto$



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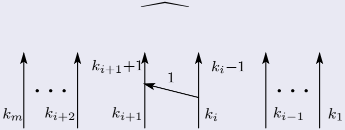
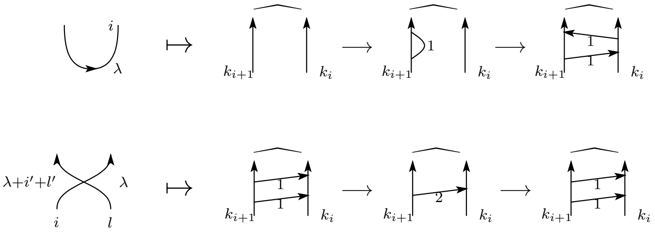


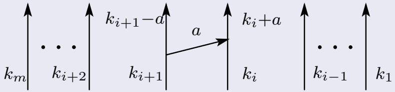
On 2-morphisms, e.g.



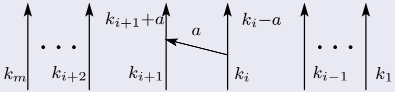
Ladders

Let $m = d = N\ell$ and $\Lambda = N\omega_\ell$. Recall

$$E_{+i}^{(a)} 1_\lambda \quad \mapsto$$



$$E_{-i}^{(a)} 1_\lambda \quad \mapsto$$



Definition

We call these webs N*-ladders with* m *uprights*.

The web category

Definition (The web category $\mathcal{W}^{\circ}(\vec{k}, N)$)

Objects: formal direct sums of N-ladders with m uprights in $W(\vec{k}, N)$.

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Let $\dot{\mathcal{W}}^\circ(\vec{k}, N)$ be the Karoubi envelope.

Definition (M.M.-Yonezawa)

Define

$$\mathcal{W}^{\circ}_{\Lambda} := \bigoplus_{\vec{k} \in \Lambda(m,m)_N} \mathcal{W}^{\circ}(\vec{k}, N).$$

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Theorem (M.M-Yonezawa)

The 2*-functor* $\Gamma_{m,m,N}$ *induces a strong* $\mathfrak{sl}_m$ 2*-representation on* $\dot{\mathcal{W}}^{\circ}_{\Lambda}$*, which is equivalent to* $\mathcal{V}^{p}_{\Lambda}$*.*

The categorification proposition

Definition (M.M.-Yonezawa)

Define the linear map

$$\psi_{\vec{k},N} \colon W(\vec{k},N) \to K_0^q(\dot{\mathcal{W}}^\circ(\vec{k},N))$$

by

$$u \mapsto [u]$$

for any N-ladder with m-uprights $u \in W(\vec{k},N)$.

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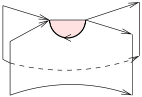
Proposition (M.M.-Yonezawa)

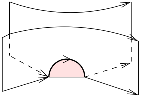
The map $\psi_{\vec{k},N}$ is an isomorphism.

Moreover, it intertwines the q-sesquilinear web form and the Euler form.

The case $N = 3$ (joint with Pan and Tubbenhauer)

Consider formal $\mathbb{C}$-linear combinations of isotopy classes of singular cobordisms, e.g. the *zip* and *unzip*:





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We also allow dots, which cannot cross singular arcs.

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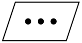
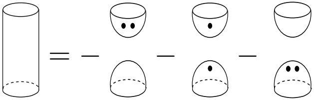
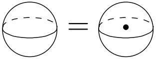
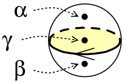
Mod out by the ideal generated by $\ell = (3D, NC, S, \Theta)$ and the *closure relation*:

Khovanov's local relations: $\ell = (3D, NC, S, \Theta)$

 $= 0$ (2)

 (3)

 $= 0,$  $= -1$ (4)



$$= \begin{cases} 1 & (\alpha, \beta, \gamma) = (1, 2, 0) \text{ or a cyclic permutation} \\ -1 & (\alpha, \beta, \gamma) = (2, 1, 0) \text{ or a cyclic permutation} \\ 0 & \text{else} \end{cases}$$

(5)

The relations in ℓ suffice to evaluate any closed foam!

The category of foams

Definition

Let Foam_3 be the graded category of webs and foams.

The *q-grading* of a foam U is defined as

$$q(U) := \chi(\partial U) - 2\chi(U) + 2d + b.$$

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Theorem (M.M.-Vaz)

There exists a fully faithful functor

$$\mathrm{Foam}_3 \to \mathrm{HMF}_3.$$

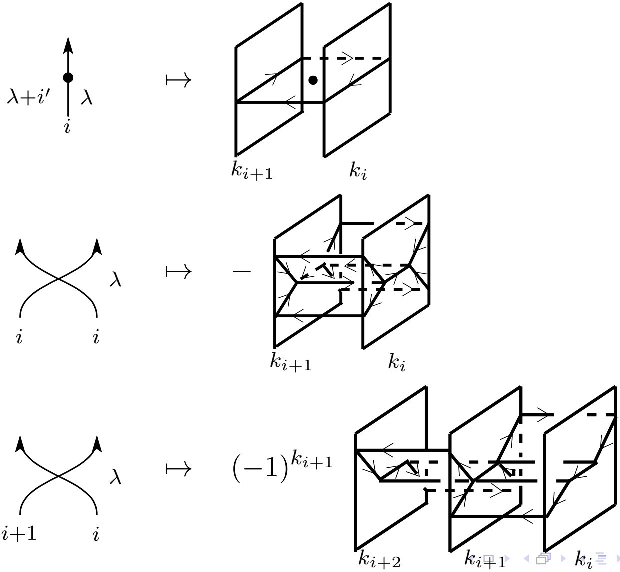
In particular, we have

$$\mathrm{Foam}_3(u,v) \cong \mathrm{Ext}(\hat{u},\hat{v}).$$

for any two monomial $\mathfrak{sl}_3$ webs u, v.

The $\mathfrak{sl}_m$ 2-representation, e.g.

Warning: facets labeled 0 or 3 have to be removed.



A good question

How to define $\mathfrak{sl}_N$ foams in general?

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So far M.M, Stošić and Vaz only did the special case of $\mathfrak{sl}_N$-foams with facets colored 1,2 and 3.

The End

THANKS!!!