André - notes for lectures 3 and 4

Lecture 3

The curve with shortest length may not be unique (e.g. antipodal points on a round sphere)

In "normal circumstances" one can show that there is indeed a shortest curve

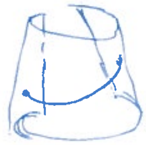
e.g in the plane \ origin there is no curve realizing distance 2 between the points $(-1,0)$ and $(1,0)$.

If "these things" do not happen then there is always a shortest curve

Slide 1

Example of isometry:

flat sheet of paper



warp the sheet

distance is still the same.

We change the embedding in space but not the intrinsic properties

Can't fold it otherwise the distance would change (the map would not be a diffeomorphism)

The two above are not related by a rigid motion

there are way more isometries than rigid motions.

With the above definition it is not practical to check whether a map is an isometry

Slide 2

To check that the definitions of isometry correspond use

$$\text{length}(\gamma) = \int_{t_0}^{t_1} \sqrt{\langle \dot\gamma(t), \dot\gamma(t)\rangle}\, dt \qquad \gamma : [t_0, t_1] \to S \subset \mathbb{R}^3$$

To check the converse you would need the exponential map.

Comment about Gauss' Theorema Egregium

- Egregium = "awesome" (Portugal is the only country where the word egregium appears in the national anthem (we say our grandparents are awesome))
- S surface in $\mathbb{R}^3$ $\rightsquigarrow$ Gauss map $N : S \to \mathbb{R}^3$ $\rightsquigarrow$ $dN : T_pS \to T_pS$ $\rightsquigarrow$ $K = \lambda_1\lambda_2$

The Gauss map records how the surface is embedded in space (it's certainly different [illegible] the above examples)

One would expect the Gaussian curvature to depend on how the surface embeds in space. It is amazing that this depends only on the intrinsic distances. There is no reason for this to be true a priori.

This was such a game changer that people started to take the intrinsic point of view which is now the usual one when one learns Riemannian geometry.

Smooth is necessary! Gauss probably did not include this. But Nash-Kuiper found C^1-isometries of a flat torus in $\mathbb{R}^3$!

Idea: crumple it. The image is not smooth (it has lots of kinks, but this can be made C^1, indeed $C^{1,\alpha}$ for $\alpha < 1/3$.

There could not be a C^2-isometry - Gaussian curvature would be 0 but this can't be for a compact surface

Indeed ~~also~~ C^1 isometries are not rigid, they can approximate any continuous map.

Gauss certainly did not know this

$$g = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$$

slide 3

Gauss observed that K depends only on g and certain of its derivatives (a priori it should depend on the quadratic form corresponding to dN) therefore it is invariant under isometries

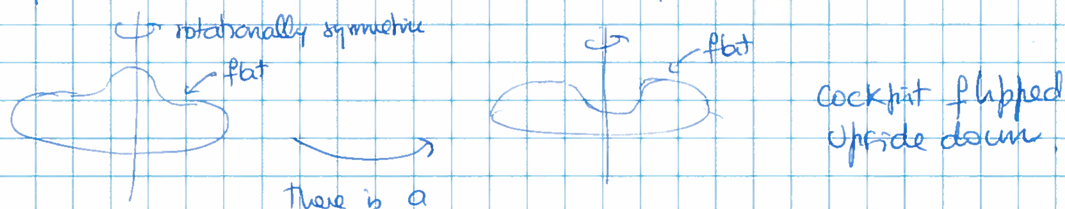
Everything is connected

Note that rigidity does not hold for the surfaces of slide 1 which have $K=0$ but these are not compact. So compactness is essential. There are also annuli with $K=1$.

slide 4

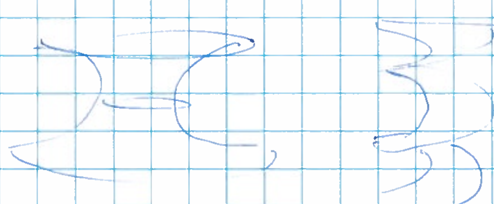
Positivity is also essential in Cohn-Vossen's slightly more complicated Theorem.

Example of compact isometric surfaces that do not differ by rigid motion



There is a distance preserving map given by gluing identity and reflection

They do not differ by rigid motion.



catenoid

helicoid

simplest minimal surfaces

can't go from one to the other by rigid motions but they are locally isometric

(They are not compact otherwise this would solve the long standing open problem)

$\bar{K} = k_1 \cdot k_2$ with $k_1 \leq k_2$ so $k_1 \leq \sqrt{\bar{K}} \leq k_2$ slide 5

__Point 2__ is where compactness comes into play

totally umbilical $\Rightarrow$ plane or sphere and the first is not possible by compactness

__Pavlov reflex__ : function with a minimum . Compute the Hessian .

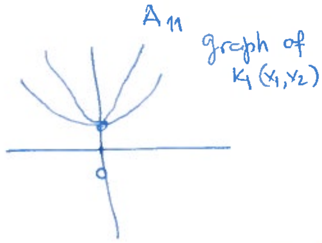
A = second fundamental form

__Point 3__ : Can without loss assume further that $\frac{\partial \bar{x}}{\partial x_1}(0) = e_1$, $\frac{\partial \bar{x}}{\partial x_2}(0) = e_2$

bold x

$A(e_1, e_1) = k_1$
$A(e_1, e_2) = 0$
$A(e_2, e_2) = k_2$

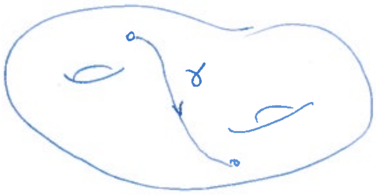
(pick appropriate coordinates at the tangent plane at the point)



$k_1 \leq A_{11}$ because of the variational characterization of the first eigenvalue (exercise)

$\partial^2_{x_2 x_2} A_{11}(0) = \partial^2_{x_1 x_1} A_{22}(0) - K(0)(k_2(0) - k_1(0))$ $\Rightarrow k_2(0) = k_1(0)$.

slide 6

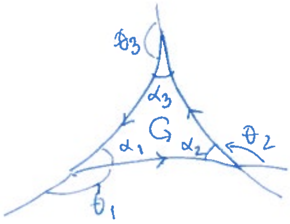


compactness is crucial otherwise the disk minus the origin is a counterexample (Think of antipodal points)

Compactness may be replaced by completeness

S

triangle = 3 points connected by 3 curves



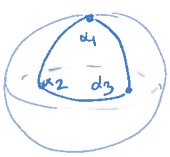
α_i = angle between tangent ~~points~~ vectors at the point.

For the external angles need to orient the curves

Note $\alpha_i + \theta_i = \pi$

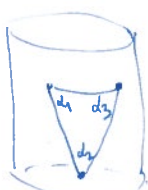
slide 7

Positive curvature $K > 0$



$\sum_{i=1}^{3} \alpha_i > \pi$

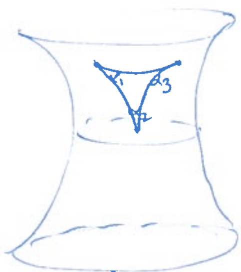
$K = 0$



$\sum_{i=1}^{3} \alpha_i = \pi$

(Thales' Theorem)

$K < 0$



$\sum_{i=1}^{3} \alpha_i < \pi$

TÉCNICO LISBOA

Gauss observed that this formula interprets curvature as ~~the defect~~ measuring the deviation from Thales' theorem.

slide 7

Local Gauss-Bonnet Want to use the best possible coordinates

without loss of generality $g_{11} = g_{22} = \lambda^2$, $g_{12} = g_{21} = 0$ $\left(g_{ij} = \frac{\partial x}{\partial x_i} \cdot \frac{\partial x}{\partial x_j} \right)$

(conformal coordinates)

Then $K = -\frac{1}{\lambda^2} \Delta (\ln \lambda)$

area conversion factor for the chart x

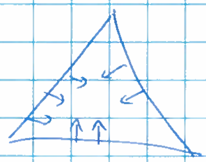
$$\int_T K \, dA = \int_T -\frac{1}{\lambda^2} \Delta \ln \lambda \; \lambda^2 \, dx_1 dx_2 = -\int_T \Delta \ln \lambda \; dx_1 dx_2$$

really λ composed with the chart

$\parallel$ Green's formula (divergence theorem)

$$\oint_{\partial T} \langle \nabla \ln \lambda, \nu \rangle$$

interior unit normal

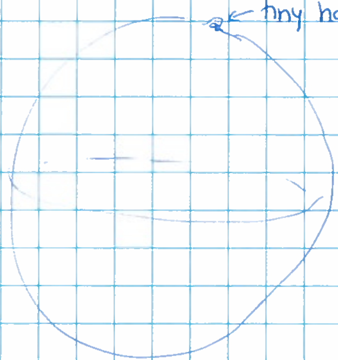


~~This then~~ ~~so~~ This line integral then needs to be interpreted in terms of the angles.

slide 8

$$\int_S K \, dA = 4\pi (1-g)$$ (Check on the round sphere of radius 1 where $K = 1$)

Suppose we have a round sphere then $\int K = 4\pi$



tiny handle

now the integral must be 0. No matter what the shape of the tiny handle the curvature will be so negative that the average is 0

The size of the handle doesn't matter of course

Theorem also implies that if $K > 0$ then the surface must be a sphere, if it is 0 must be a torus and if $K < 0$ then $g \geq 2$

This ~~sets~~ theorem sets the standard of what a great theorem in geometry is because the 2 sides live in different worlds: one geometry, the other pure topology

Best theorems in geometry: conditions on curvature $\Rightarrow$ topological conditions

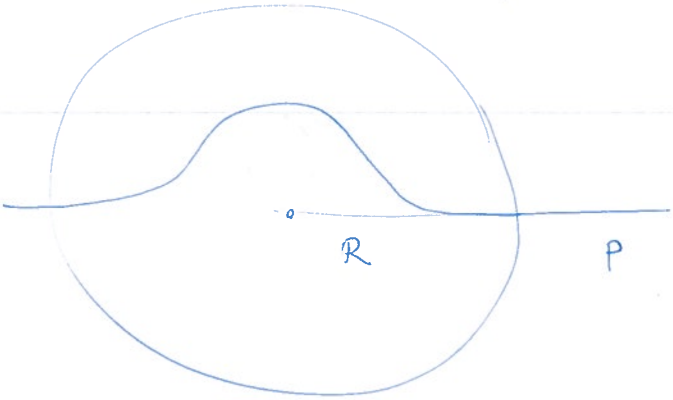
surface is
wiggled inside a ball

or more schematically 

slide 9

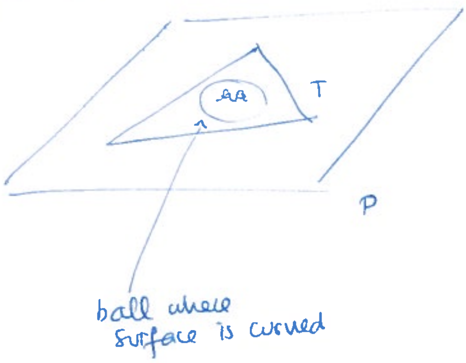
The theorem says we have to make the curvature negative somewhere

It turns out this is also true for higher dimensional manifolds. Gaussian curvature becomes scalar curvature. [proved by Witten, Chen-Yau for dims ≤ 7
now for all dimensions. These are called
positive mass theorems]



surface equals the
plane outside the ball

seen from above

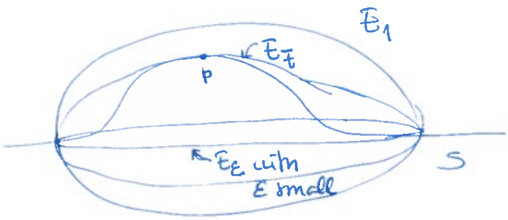


Apply Gauss-Bonnet to the triangle T

Local Gauss-Bonnet $\Rightarrow \int_T K\,dA = 0$

so $K=0$ everywhere.

This is not enough yet.



S at p must be more curved than E_t

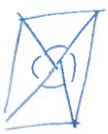
so the curvature of S at p is positive. Contradiction

slide 10

To get Euler's formula you can use induction by attaching more and
more handles and see how the number changes

slide 11

N = F because the number of faces equals the number of triangles



Sum of interior angles
at a given vertex is 2π.