Surfaces in space

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Last time

On a surface $S \subset \mathbb{R}^3$ there is notion of *intrinsic* distance

$$d(p,q) = \inf\{\text{length}\,(\gamma) | \gamma : [0,1] \to S \text{ curve with } \gamma(0) = p, \gamma(1) = q\}$$

Definition

An isometry between surfaces S_1, S_2 is a diffeomorphism $\phi : S_1 \to S_2$ that preserves distance, i.e., $d_{S_1}(p,q) = d_{S_2}(\phi(p), \phi(q))$ for all $p, q \in S_1$.

If ϕ is a rigid motion, i.e., a composition of translations, rotations, and reflections in $\mathbb{R}^3$ then ϕ is an isometry between S and $\phi(S)$.

There are many isometries that do not come from rigid motions.

Gaussian Curvature

If a diffeomorphism $\phi : S_1 \to S_2$ is such that

$$d\phi_p(v).d\phi_p(w) = v.w \quad \text{for all } v, w \in T_pS_1,\ p \in S_1$$

then ϕ is an isometry.

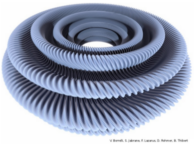
This is because if γ is a curve in S_1 then the condition above implies that length $(\phi(\gamma))$ = length (γ). (Check!)

The converse is also true.

Gauss's Theorema Egregium

The Gaussian curvature is invariant under smooth isometries: If ϕ is an isometry between S_1 and S_2 then $K^{S_2} \circ \phi = K^{S_1}$.

There is a C^1-isometry from a flat torus into $\mathbb{R}^3$



Gaussian curvature

"*The Gaussian curvature is invariant under isometries.*"

Proof.

Given a chart $(U, \mathbf{x})$ set $g_{ij} = \frac{\partial \mathbf{x}}{\partial x_i} \cdot \frac{\partial \mathbf{x}}{\partial x_j}$, $i, j = 1, 2$.

$K \circ \mathbf{x}$ can be computed via a polynomial expression that depends only on $(g_{ij})^2_{i,j=1}$, $(\partial_{x_k} g_{ij})^2_{i,j,k=1}$, and $(\partial^2_{x_k x_l} g_{ij})^2_{i,j,k,l=1}$.

For example, if $g_{11} = g_{22} = \lambda^2$ and $g_{12} = 0$, then $K = -\lambda^{-2} \Delta \ln \lambda$.

If ϕ is an isometry between S_1 and S_2, then $(U, \mathbf{y} = \phi \circ \mathbf{x})$ is a chart for S_2 and we have

$$\frac{\partial \mathbf{y}}{\partial x_i} \cdot \frac{\partial \mathbf{y}}{\partial x_j} = d\phi(\frac{\partial \mathbf{x}}{\partial x_i}).d\phi(\frac{\partial \mathbf{x}}{\partial x_j}) = \frac{\partial \mathbf{x}}{\partial x_i} \cdot \frac{\partial \mathbf{x}}{\partial x_j} = g_{ij}$$

Thus $K^{S_2} \circ \mathbf{y} = K^{S_1} \circ \mathbf{x} \implies K^{S_2} \circ \phi = K^{S_1}$.

□

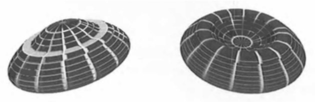
Gaussian curvature

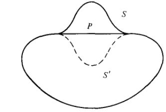
Rigidity of sphere

Compact surfaces with constant Gaussian curvature $\bar{K}$ are rigid, i.e., up to a rigid motion they are spheres with radius $1/\sqrt{\bar{K}}$.

Cohn-Vossen Theorem says that any two isometric compact surfaces with positive Gaussian curvature are rigid, i.e., differ only by a rigid motion.

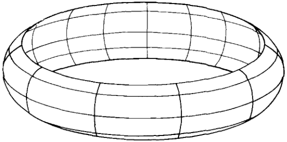
There are compact isometric surfaces that do not differ by rigid motion





It is an open problem to determine if compact surfaces are locally rigid.

The hypothesis that the surface is compact in the theorem is crucial as there are annuli with $K = 1$.



Gaussian curvature

Compact surfaces with constant Gaussian curvature $\bar{K}$ are rigid.

1: $\bar{K}$ must be positive because there is a point with positive Gaussian curvature. We must have $k_1 \le \sqrt{\bar{k}} \le k_2$.

2: Choose p maximum point of k_2. It is also minimum point of $k_1 = \bar{K}/k_2$.

If we show that S is umbilic at p then the global maximum of k_2 is also its global minimum $\sqrt{\bar{k}}$ and thus S is totally umbilical.

3: W.l.o.g. $p = 0$, $S = \{(x_1, x_2, h(x_1, x_2)) : (x_1, x_2) \in U\}$, $h(0) = 0$, $\nabla h(0) = 0$.

With $\mathbf{x}(x_1, x_2) = (x_1, x_2, h(x_1, x_2))$ set

$$A_{11} = A\left(\frac{\partial_{x_1}\mathbf{x}}{|\partial_{x_1}\mathbf{x}|}, \frac{\partial_{x_1}\mathbf{x}}{|\partial_{x_1}\mathbf{x}|}\right) \quad A_{22} = A\left(\frac{\partial_{x_2}\mathbf{x}}{|\partial_{x_2}\mathbf{x}|}, \frac{\partial_{x_2}\mathbf{x}}{|\partial_{x_2}\mathbf{x}|}\right)$$

We have $k_1 \le A_{11}$ and $k_2 \ge A_{22}$ near the origin and equality at the origin.

Thus $\partial^2_{x_1 x_1} A_{22}(0) \le 0 \le \partial^2_{x_2 x_2} A_{11}(0)$.

A long computation shows $\partial^2_{x_2 x_2} A_{11}(0) = \partial^2_{x_1 x_1} A_{22}(0) - K(0)(k_2(0) - k_1(0))$.

Thus $k_2(0) = k_1(0)$ as we wanted to show. $\square$

Gauss Bonnet Theorem

Let S be a compact surface.

(Hopf-Rinow Thm.) *For every two points in S there is curve γ in S that minimizes the intrinsic distance between the two points. The curve γ is called a length minimizing geodesic.*

A *geodesic triangle* T on S is a triangle whose edges are length minimizing geodesics.

After orienting the triangle T we have interior angles $0 \leq \alpha_i \leq 2\pi$ and exterior angles $-\pi \leq \theta_i \leq \pi$, $i = 1, 2, 3$.

Local Gauss-Bonnet Theorem

Let $T \subset S$ be a geodesic triangle contained in the image of some chart. Then

$$\int_T KdA + \sum_{i=1}^{3} \theta_i = 2\pi.$$

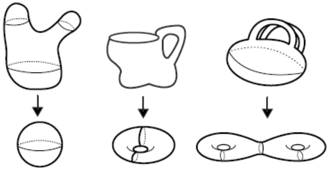
Thus, because $\alpha_i + \beta_i = \pi$ we have $\sum_{i=1}^{3} \alpha_i = \int_T KdA + \pi$.

Gauss-Bonnet Theorem

"*T geodesic triangle* $\implies \sum_{i=1}^{3} \alpha_i = \int_T KdA + \pi$."

- $K = 0 \implies$ sum of interior angles is π
- $K > 0 \implies$ sum of interior angles is $> \pi$
- $K < 0 \implies$ sum of interior angles is $< \pi$

Every compact orientable surface is homeomorphic to a sphere with *g*-handles (genus)



Gauss-Bonnet Theorem

Let S be a compact surface with genus g. Then

$$\int_S KdA = 4\pi(1-g)$$

Application

Theorem: *A surface S with $K \geq 0$ and identical to a plane P outside a compact set, is identical to P everywhere*

Proof.

1: Choose $R > 0$ so that $S \setminus B_R(0) = P \setminus B_R(0)$;

2: Choose T a triangle in P that contains $B_R(0) \cap P$ in its interior;

3: Local G.-B. Thm $\implies \int_T KdA = 0$ and so $K = 0$ everywhere;

4: Let $\{\Sigma_t\}_{0<t<\leq 1}$ be a continuous family of ellipsoids such that

1. all have positive Gaussian curvature;
2. $B_R(0) \cap P$ is contained inside $\Sigma_t \cap P$ for all t;
3. $B_R(0)$ is contained inside Σ_1 and for all t very small , Σ_t is contained inside the slab $\{(x, y, z) : |z| < t\}$.

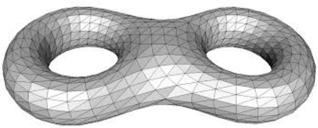
5: Initially $S \cap P \subset \Sigma_1$. Decrease t from 1 to 0 until Σ_t touches for the first time S at some point p and instant $\bar{t}$.

6: S is inside Σ_t near p and so $K^S(p) \geq K^{\Sigma_{\bar{t}}}(p) > 0$. Contradiction

□

Gauss-Bonnet Theorem

A triangulation $\{T_i\}_{i=1}^N$ of S is a collection of triangles so that $\cup_{i=1}^N T_i = S$ and triangles intersect only along edges or vertices.



The Euler characteristic χ is $\mathbf{V}\mathit{ertices} - \mathbf{E}\mathit{dges} + \mathbf{F}\mathit{aces}$ and does not depend on the triangulation.

For a compact surface with genus g, $\chi = 2(1 - g)$.

Every surface admits a triangulation made by geodesic triangles contained in image of charts.

Gauss-Bonnet Theorem

$$S \text{ be a compact surface with genus } g \implies \int_S K dA = 4\pi(1-g).$$

Proof.

Let $\{T_i\}_{i=1}^N$ be a geodesic triangulation. Then

$$\sum_{i=1}^N \int_{T_i} K dA + \sum_{i=1}^N \sum_{j=1}^3 \theta_{ij} = 2\pi F$$

Using $\theta_{ij} = \pi - \alpha_{ij}$ we have

$$\int_S K dA + 3\pi F - \sum_{i=1}^N \sum_{j=1}^3 \alpha_{ij} = 2\pi F$$

Every edge is shared by two triangles and so $3F = 2E$

$$\int_S K dA + 2\pi E - \sum_{i=1}^N \sum_{j=1}^3 \alpha_{ij} = 2\pi F.$$

The sum of all interior angles around a vertice is 2π and so the sum of all interior angles in $2\pi V$.Thus

$$\int_S K dA + 2\pi E - 2\pi V = 2\pi F \implies \int_S K dA = 2\pi(V - E + F) = 4\pi(1-g)$$

□