TÉCNICO LISBOA

Notes for lecture 5 - André Neves

The topological formula in Gauss-Bonnet is great on one hand but if we are looking to understand how the surface is embedded in space it doesn't help at all. slide

To get at this we can use the Willmore energy.

Why $|H|^2$? We want the quantity to be scale invariant (~~expect to~~ does not change when we scale the ~~____~~ surface):

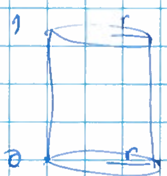
$$W(\lambda S) = W(S) \quad \text{for } \lambda \in (0,+\infty)$$

The square makes the integral scale invariant

$H(S(r)) = \frac{1}{r} + \frac{1}{r} = 2/r$

sphere of radius r

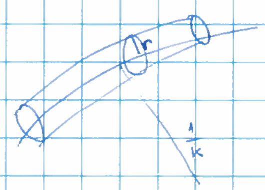
A genus 0 surface with large Willmore energy

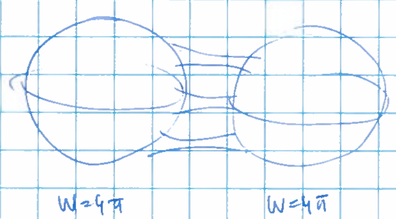


$$W(cyl) = \frac{1}{4}\int_{Cyl} \frac{1}{r^2}\, dA = \frac{1}{4}\,\frac{2\pi r}{r^2} = \frac{\pi}{2r}$$

as $r \to 0$, $W(cyl_r) \to \infty$. Capping the cylinder off we get a genus 0 surface.

~~If we have a~~ slide 2





If the cylinders are too long compared with radius ~~then~~ otherwise the Willmore energy will blow up.

The example is the stereographic projection of the Lawson minimal surface.
The spheres are connected by catenoids.

In Willmore's theorem compactness is essential (a plane has Willmore energy 0) slide 3

~~For~~ the area formula, if N is a diffeomorphism, is just the change of variable formula.

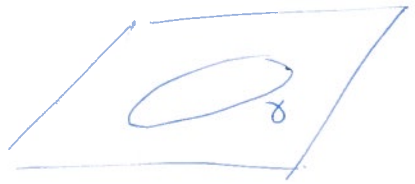
Start with PLV very far and move it until it touches the surface for the first time. If p is this point then $N(p) = v$ and ξ in ~~the~~ problem set 3 you saw the Gaussian curvature must be non-negative.

$v = (0,0,-1)$ want it to be in the image

~~move the plane up~~

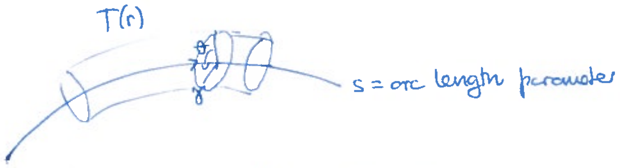
slide 4

The Fenchel inequality ~~...~~ was what ~~...~~ inspired Willmore's result.



(recall $k = |\ddot{x}(s)|$ where s is arc length)

We'll deduce the inequality from Willmore's theorem but ~~...~~ this is anachronistic.



curvature of the curve

$$K\,dA = -k\cos\theta\, ds\, d\theta \quad \text{(check)}$$

Note that for each s, K will be positive for half the interval $[0, 2\pi]$ and negative in the other half

For the ~~reverse inequality~~ conclusion in case of equality see do Carmo..

Note: If the curve is not embedded wiggle it a little bit so that it becomes embedded. That will not change $\int k\,ds$

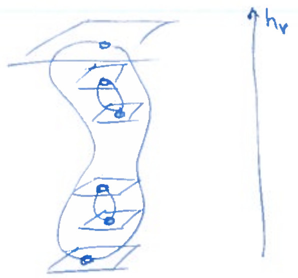
slide 5

The proof of (Langevin-Rosenberg) uses two sets of ideas.

Chern-Lashof

Fary-Milnor ~~...~~

$V = (0,0,1)$, $h_V(x,y,z) = z$ is just the usual height function



$$\int_{S^2} \#\,\mathrm{crit}(h_V) = \int_{S^2} \# N^{-1}(v) + \int_{S^2} \# N^{-1}(-v) = 2\int_{S^2} \# N^{-1}(v)$$

They are the same because the antipodal map is an isometry of the sphere

If the torus is not knotted then there are at least 4 critical points.

~~...~~ If a curve is knotted there are at least 4 critical points

The Milno-Fory theorem can be deduced from Langmin–Rosenberg in the same way as before. But this is again anachronustic

slid 6

Conjecture: the surface with the least Wilmore energy is the Lawson ~~[illegible]~~ genus g surface

There is no ~~[illegible]~~ explicit parametrization for these surfaces. They are constructed by solving a Plateau problem and exploiting symmetry.