Surfaces in space

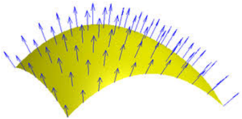
André Neves

THE UNIVERSITY OF CHICAGO

Last time

Given an orientable surface S we have the Gauss map

$$N : S \to S^2 \subset \mathbb{R}^3, \quad N(p) = \text{unit normal vector to } T_pS.$$



Its differential satisfies $dN_p : T_pS \to T_pS$. and one can check that dN_p is self-adjoint.

Alternatively, we can also consider the 2^{nd} fundamental form

$$A : T_pS \times T_pS \to \mathbb{R}, \quad A(v, w) = -dN_p(v).w.$$

In the (not) particular case where

$$S = \{(x_1, x_2, f(x_1, x_2)) : (x_1, x_2) \in U\} \quad f(0) = 0, \nabla f(0) = 0$$

we have $A_0 = \text{Hess}\, f$.

Principal curvatures

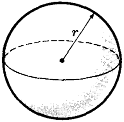
The map dN_p is self-adjoint and thus diagonalizable, i.e., there is $\{e_1, e_2\}$ an orthonormal basis for T_pS so that

$$dN_p(e_i) + k_i(p)e_i = 0, \quad i = 1, 2, \quad \text{where } k_1(p) \leq k_2(p).$$

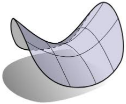
The functions k_1, k_2 are continuous and called *principal curvatures.* The directions e_1, e_2 are called *principal directions.*

On $P = \{z = 0\}$, $N_p = (0, 0, 1)$ all p $\implies dN_p = 0 \implies k_1 = k_2 = 0$

On $S(r) = \{p \in \mathbb{R}^3 : |p| = r\}$, $dN_p = -r^{-1}\mathrm{Id} \implies k_1 = k_2 = 1/r$

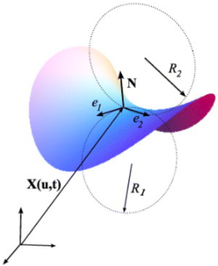


On $S = \{z = x^2 - y^2\}$ the principal curvatures at the origin are -2 and 2.



Principal curvatures

Locally, a surface whose principal curvatures are $k_1 = -1/R_1$ and $k_2 = 1/R_2$ at a point p looks like the one given below.



Umbilic points

We say a point $p \in S$ is *umbilic* if $k_1(p) = k_2(p)$ and the surface is totally umbilical if every point is umbilic.

Theorem

A connected totally umbilical surface is contained in a plane or sphere.

If the principal curvatures are constant functions, one can show that S is contained in a plane, sphere, or cylinder.

If a compact surface has no umbilic points, then the principal directions give a non-vanishing vector field on the surface. Poincare-Hopf index Theorem says that the surface must have genus one.

It is an old problem (Caratheodory Conjecture) to show that every sphere has at least two umbilic points.

Umbilic points

"Totally umbilic surfaces are contained in planes or spheres"

Proof.

We have for all p that $dN_p = \lambda(p)\mathrm{Id}$. Let $(U, \mathbf{x})$ be a chart.

1: *The continuous function $\bar{\lambda} = \lambda \circ \mathbf{x}$ is constant ($\implies \lambda = const.$)*

$dN(\frac{\partial \mathbf{x}}{\partial x_1}) = \bar{\lambda}\frac{\partial \mathbf{x}}{\partial x_1} \implies \frac{\partial(N \circ \mathbf{x})}{\partial x_1} = \bar{\lambda}\frac{\partial \mathbf{x}}{\partial x_1} \implies \frac{\partial^2(N \circ \mathbf{x})}{\partial x_2 \partial x_1} = \frac{\partial \bar{\lambda}}{\partial x_2}\frac{\partial \mathbf{x}}{\partial x_1} + \bar{\lambda}\frac{\partial \mathbf{x}}{\partial x_2 \partial x_1}.$

Switching x_1 with x_2, $\frac{\partial^2(N \circ \mathbf{x})}{\partial x_1 \partial x_2} = \frac{\partial \bar{\lambda}}{\partial x_1}\frac{\partial \mathbf{x}}{\partial x_2} + \bar{\lambda}\frac{\partial \mathbf{x}}{\partial x_1 \partial x_2}.$

Thus $\frac{\partial \bar{\lambda}}{\partial x_1}\frac{\partial \mathbf{x}}{\partial x_2} = \frac{\partial \bar{\lambda}}{\partial x_2}\frac{\partial \mathbf{x}}{\partial x_1} \implies \frac{\partial \bar{\lambda}}{\partial x_1} = \frac{\bar{\lambda}}{\partial x_2} = 0.$

2: If $\lambda = 0 \implies dN_p = 0$ all $p \implies N(p) = const. \implies S \subset$ some plane.

3: If $\lambda > 0 \implies \frac{\partial \mathbf{x}}{\partial x_i} - \frac{1}{\lambda}\frac{\partial(N \circ \mathbf{x})}{\partial x_i} = 0 \implies \frac{\partial}{\partial x_i}(\mathbf{x} - \frac{1}{\lambda}N \circ \mathbf{x}) = 0, i = 1, 2$

Thus $\exists c \in \mathbb{R}^3$ such that $\mathbf{x} - \frac{1}{\lambda}N \circ \mathbf{x} = c \implies |\mathbf{x} - c| = \frac{|N \circ \mathbf{x}|}{\lambda} = \lambda^{-1}.$ □

Gaussian curvature and mean curvature

Definition

The Gaussian curvature of a surface S at p is $K(p) = det\, dN_p = k_1(p)k_2(p)$.

The mean curvature of an orientable surface S at p is $H(p) = k_1(p) + k_2(p)$.

For $S(r) = \{p \in \mathbb{R}^3 : |p| = r\}$ we have $K = r^{-2}$ and $H = 2r^{-1}$.

Claim: *The functions* $K : S \to \mathbb{R}$ *and* $H : S \to \mathbb{R}$ *are smooth.*

If S is given by the graph of a function $h : U \subset \mathbb{R}^2 \to \mathbb{R}$, then

$$K = \frac{\partial^2_{xx} h \partial^2_{yy} h - (\partial^2_{xy} h)^2}{(1 + |\nabla h|^2)^2} = \frac{\det \mathrm{Hess} h}{(1 + |\nabla h|^2)^2}$$

and

$$H = \frac{(1 + (\partial_x h)^2)\partial^2_{yy} h - 2\partial_x h \partial_y h \partial^2_{xy} h + (1 + (\partial_x h)^2)\partial^2_{yy} h}{(1 + |\nabla h|^2)^2}.$$

Thus K and H are smooth.

Gaussian curvature

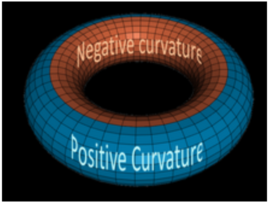
Suppose $S = \{(x, y, h(x, y)) : (x, y) \in U\}$, where $h(0) = 0, \nabla h(0) = 0$.

Then $K(0) = \det \text{Hess} h(0)$.

If $K(0) > 0$, h has a local min or a local max at 0 by the 2^{nd} derivative test.

If $K(0) < 0$, h has a saddle point at 0 by the 2^{nd} derivative test.

In general, if $K(p) > 0$ then the surface near p is all to one side of T_pS while if $K(p) < 0$ then the surface near p is on both sides of T_pS.



Gaussian curvature

Proposition

If S is a compact surface, then there is a point p with $K(p) > 0$.

Proof.

There is $R > 0$ so that $S \subset B_R(0)$. Decrease R until we find r so that $S \subseteq B_r(0)$ and $S \cap \partial B_r(0) \neq \emptyset$.

At the point $p \in S \cap \partial B_r(0)$ we must have $K^S(p) \geq K^{\partial B_r(0)}(p) = r^{-2}$. □

On a surface $S \subset \mathbb{R}^3$ there is notion of *intrinsic* distance

$$d(p,q) = \inf\{\text{length}\,(\gamma) | \gamma : [0,1] \to S \text{ curve with } \gamma(0) = p, \gamma(1) = q\}$$

On the unit sphere $d((0,0,1),(0,0,-1)) = \pi$.

Definition

An isometry between surfaces S_1, S_2 is a diffeomorphism $\phi : S_1 \to S_2$ that preserves distance, i.e., $d_{S_1}(p,q) = d_{S_2}(\phi(p),\phi(q))$ for all $p, q \in S_1$.