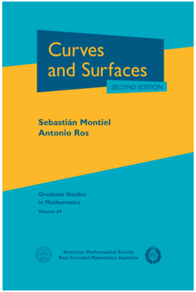
Surfaces in space

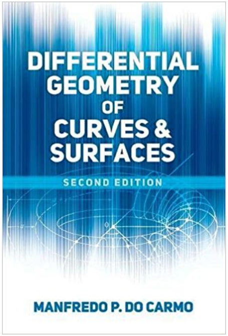
André Neves

THE UNIVERSITY OF CHICAGO

Texbooks

- Curves and Surfaces 2^{nd} ed, Montiel-Ros, GSM Vol 69, AMS
- Differential Geometry of Curves and Surfaces, do Carmo, Dover Books on Mathematics





Overview

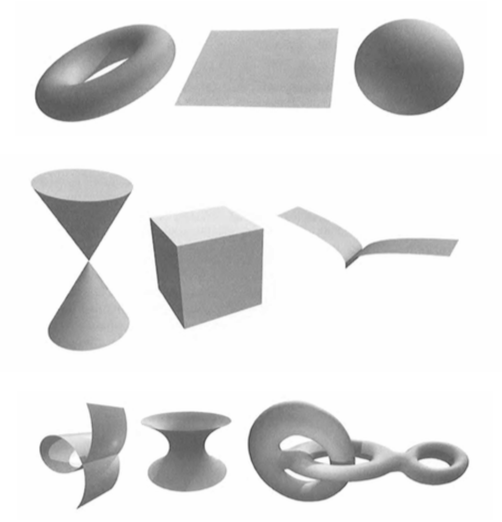
Given a surface in space we want to (i) associate a quantity to that surface so that (ii) information about that quantity tells something non-trivial about the surface.

The most obvious one, the area, is not good enough because we can always just by scaling making its area small or big...

We will study two quantities, Gaussian curvature and mean curvature, that have been introduced around 200 years ago and have proven to be very efficient in answering these type of questions.

What is a surface?

" A surface is a subset of $\mathbb{R}^3$ such that each of its points has a tiny neighborhood that looks like a curved disc."



What is a surface?

Definition

A set $S \subset \mathbb{R}^3$ is a surface if for every $p \in S$ there are open sets $V \subset \mathbb{R}^3$ and $U \subset \mathbb{R}^2$ and map

$$\mathbf{x} : U \subset \mathbb{R}^2 \longrightarrow V \cap S \subset \mathbb{R}^3$$

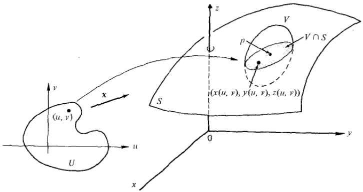
such that

i) *The map $\mathbf{x}$ is smooth;*

ii) *$\mathbf{x}(U) = V \cap S$ and $\mathbf{x}$ is a homeomorphism;*

iii) *For all $q \in U$, the linear map $d\mathbf{x}_q : \mathbb{R}^2 \to \mathbb{R}^3$ is injective.*

The pair $(\mathbf{x}, U)$ is called a chart.



Condition (iii) ensures that a cone is not a surface.

Differentiability

Examples

- A plane $P \subset \mathbb{R}^3$ or any open set of P are surfaces;
- If $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ is smooth then its graph is a surface

$$graph(f) = \{(u, v, f(u, v)) : (u, v) \in U\};$$

- If $f : \mathbb{R}^3 \to \mathbb{R}$ a smooth function such that $t \in \mathbb{R}$ is a regular value, i.e., $\nabla f(p) \neq 0$ for all $p \in f^{-1}(t)$, then $S = f^{-1}(t)$ is a surface (called level surface).

Definition

Let S be a surface and Ω be an open set of some $\mathbb{R}^k$.

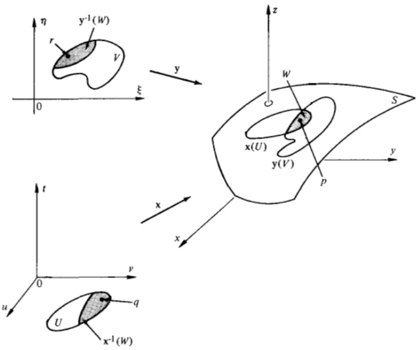
- *A function $f : \Omega \subset \mathbb{R}^k \to S \subset \mathbb{R}^3$ is differentiable if its 3 components are differentiable;*
- *A function $f : S \to \Omega \subset \mathbb{R}^k$ is differentiable if $f \circ \mathbf{x} : U \to \Omega$ is differentiable for every chart $(\mathbf{x}, U)$.*

Change of parameters

Given two charts $(\mathbf{x}, U)$, $(\mathbf{y}, V)$ of the surface S such that $\mathbf{x}(U) \cap \mathbf{y}(V) = W \neq \emptyset$, then

$$h = \mathbf{y}^{-1} \circ \mathbf{x} : \mathbf{x}^{-1}(W) \to \mathbf{y}^{-1}(W)$$

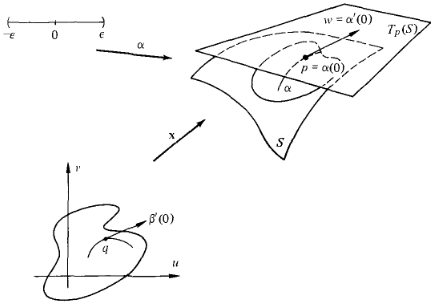
is a diffeomorphism (i..e, smooth with inverse also smooth)



Tangent plane

Definition

The tangent plane T_pS to a surface S at the point $p \in S$ is the set of all vectors $w \in \mathbb{R}^3$ for which there is a differentiable curve $\alpha : (-\varepsilon, \varepsilon) \to S$ with $\alpha(0) = p$ and $\alpha'(0) = w$.



Given a chart $(U, \mathbf{x})$ where $\mathbf{x}(q) = p$, then $T_pS = d\mathbf{x}_q(\mathbb{R}^2)$.

Tangent plane

Given a surface S and $p \in S$ there is a neighborhood $V \subset \mathbb{R}^3$ of p so that $V \cap S$ is the graph of a function defined over T_pS.

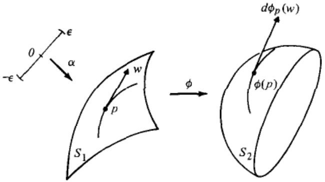
Assuming w.l.o.g. that $T_pS = \{z = 0\}$, this means there is $U \subset \mathbb{R}^2$ containing 0 and $h : U \to \mathbb{R}$ so that

$$S \cap V = \{(x, y, h(x, y)) : (x, y) \in U\}.$$

Necessarily $h(0) = 0$ and $\nabla h(0) = 0$.

Given a differentiable map $\phi : S_1 \to S_2$ between two surfaces we define

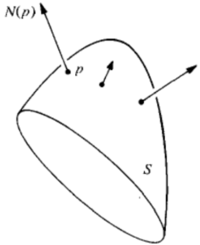
$$d\phi_p : T_pS_1 \to T_{\phi(p)}S_2, \quad d\phi_p(\alpha'(0)) = (\phi \circ \alpha)'(0)$$



Normal vector

Given a chart $(U, \mathbf{x})$, a normal vector to the surface at $p = \mathbf{x}(u, v)$ is given by

$$N(p) = \frac{\partial_u \mathbf{x} \times \partial_v \mathbf{x}}{|\partial_u \mathbf{x} \times \partial_v \mathbf{x}|}$$

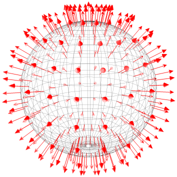


If $S = f^{-1}(t)$ for some $f : \mathbb{R}^3 \to \mathbb{R}$, where t is regular value, then

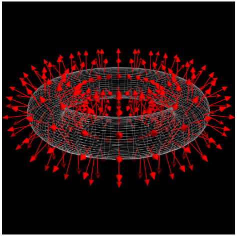
$$N(p) = \frac{\nabla f(p)}{|\nabla f(p)|} \quad \text{and} \quad T_p S = \{w \in \mathbb{R}^3 : w.\nabla f(p) = 0\}.$$

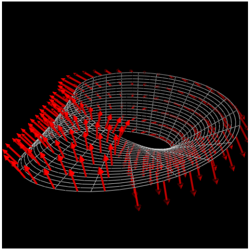
Orientable surfaces

When the surface is $S^2 = \{x^2 + y^2 + z^2 = 1\}$ (unit sphere) then $N(p) = p$ for all $p \in S^2$ and $T_pS^2 = \{w \in \mathbb{R}^3 : w.p = 0\}$



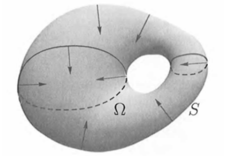
A surface S is orientable if there is a continuous map $N : S \to S^2 \subset \mathbb{R}^3$ such that $N(p)$ is a normal vector field.





Orientable surfaces

Surfaces with a well defined “inside” and “outside” are orientable. Brower separation Theorem says that any compact surface in $\mathbb{R}^3$ has an “inside” and “outside”.

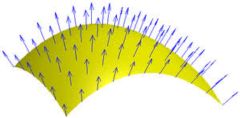


Level surfaces $S = f^{-1}(t)$ are orientable because there’s an “inside” $\{f < t\}$ and “outside” $\{f > t\}$. The converse is also true.

Gauss map

Given an orientable surface S it is a good idea to study its Gauss map

$$N : S \to S^2 \subset \mathbb{R}^3, \quad N(p) = \text{unit normal vector to } T_pS.$$



For instance, if $N(S)$ is contained in the upper hemisphere of S^2, then it is "easy" to see that S is graphical over the xy-plane.

Note that $T_{N(p)}S^2 = \{w \in \mathbb{R}^3 : w.N(p) = 0\} = T_pS$. Thus

$$dN_p : T_pS \to T_pS.$$

It is "easy" to see that dN_p is self-adjoint, i.e., $dN_p(v).w = dN_p(w).v$ for all $v, w \in T_pS$.

Second fundamental form

The 2^{nd} fundamental form at $p \in S$ is the bilinear symmetric map

$$A : T_pS \times T_pS \to \mathbb{R}, \quad A(v, w) = -dN_p(v).w.$$

The $-$ sign is so that if $S = S^2$ and N the *interior* unit normal, then $A(v, w) = v.w$.

Assuming $0 \in S$ and $T_0S = \{z = 0\}$, there is a $f : U \to \mathbb{R}$ so that near the origin, S is given by

$$\text{graph}(f) = \{(x_1, x_2, f(x_1, x_2)) : (x_1, x_2) \in U\} \quad f(0) = 0,\ \nabla f(0) = 0.$$

In this case with $\partial_{x_1} = (1, 0, \partial_{x_1} f)$ and $\partial_{x_2} = (0, 1, \partial_{x_2} f)$ we have

$$T_{(x_1,x_2,f(x_1,x_2))}S = \text{span}\{\partial_{x_1}, \partial_{x_2}\} \quad \text{and} \quad N(x_1, x_2) = \frac{(-\partial_{x_1} f, -\partial f_{x_2}, 1)}{\sqrt{1+|\nabla f|^2}}$$

Using the fact that f and ∇f vanish at the origin we compute

$$dN_0(\partial_{x_i}) = \partial_{x_i} N(0) = -(\partial^2_{x_i x_1} f(0), \partial^2_{x_i x_2} f(0), 0),\ i = 1, 2 \text{ and}$$

$$A(\partial_{x_i}, \partial_{x_j}) = -dN_0(\partial_{x_i}).\partial_{x_j} = \partial^2_{x_i x_j} f(0),\ i, j = 1, 2.$$

In other words, A is like the “Hessian” of the surface