An introduction to Lie groups, symmetries, and symplectic geometry
Problem Set 4

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Exercise 1. *Let n be a positive integer. Show that all non-degenerate anti-symmetric forms on n-dimensional vector spaces are equivalent over $\mathbb{R}$; i.e. if (V_1, β_1) and (V_2, β_2) are two such then there is a linear isomorphism $\varphi : V \to W$ such that*

$$\beta_2(\varphi(v), \varphi(w)) = \beta_1(v, w) \qquad\qquad v, w \in V_1.$$

Exercise 2. *The* Lie derivative $\mathcal{L}_x\beta$ *is* 0 *everywhere if and only $\phi_t^*\beta = \beta$ for all $t \in \mathbb{R}$.*

Exercise 3. *Let $x : U \to \mathbb{R}^n$ be a coordinate chart on a n-dimensional smooth manifold and consider the induced map*

$$(x, v) : TU \to \mathbb{R}^n \times \mathbb{R}^n$$
$$\left(p, v^i\left(\frac{\partial}{\partial x^i}\right)_p\right) \mapsto \left(x^1(p), \cdots, x^n(p), v^1, \ldots, v^n\right).$$

Let X be the vector field on U given by $X = a^i(x)\frac{\partial}{\partial x^i}$. In the coordinate chart given by (x, v) the derivative of X is given by

$$X' = a^i(x)\frac{\partial}{\partial x^i} + v^i \frac{\partial a_j}{\partial x^i}\frac{\partial}{\partial v^j}.$$

Exercise 4. *Let M be a smooth manifold and $L : TM \to \mathbb{R}$ be a smooth function. For the function $E_L : M \to \mathbb{R}$ and 1-form ω_L on TM such that in local coordinates (x, v) we have*

$$E_L = v^i \frac{\partial L}{\partial v^i} - L$$
$$\omega_l = \frac{\partial L}{\partial v^i} dx^i.$$

Check that

$$\ddot{\gamma}(t) \lrcorner d\omega_L = -(dE_L)_{\dot{\gamma}(t)}$$

expands in local coordinates to the Euler-Lagrange equations for $\gamma : [a, b] \to M$ to be L-critical.