PROBLEMS

1: Let $S_2(\mathbb{R}^2)$ be the set of all symmetric 2×2 matrices and consider a smooth map $A : \mathbb{R}^2 \to S_2(\mathbb{R}^2)$.

- If $\lambda_1(x, y)$ denotes the lowest eigenvalue of $A(x, y)$, show that the map $(x, y) \mapsto \lambda_1(x, y)$ is continuous.
- Give an example of a map A as above where the function $(x, y) \mapsto \lambda_1(x, y)$ is NOT differentiable.
- Show that the functions

$$det : \mathbb{R}^2 \to \mathbb{R}, \quad (x, y) \mapsto detA(x, y)$$

  and

$$tr : \mathbb{R}^2 \to \mathbb{R}, \quad (x, y) \mapsto traceA(x, y)$$

  are smooth.

2: Show that on $S(r) = \{p \in \mathbb{R}^3 : |p| = r\}$ we have $dN_p = -r^{-1}\mathrm{Id}$.

3: Compute the principal curvatures of the cylinder

$$\{(x, y, z) : x^2 + y^2 = 1\}.$$

4: Show that if S is a surface such that the Gauss map is constant, then S is contained in a plane

5:Show that if $S = (x, y, h(x, y)) : (x, y) \in U$ where $h : U \to \mathbb{R}$ is a smooth function then

$$K = \frac{\det \operatorname{Hess} h}{(1 + |\nabla h|^2)^2}$$

and

$$H = \frac{(1 + (\partial_x h)^2)\partial^2_{yy} h - 2\partial_x h \partial_y h \partial^2_{xy} h + (1 + (\partial_x h)^2)\partial^2_{yy} h}{(1 + |\nabla h|^2)^2}.$$

6: Give an example of two surfaces S_1, S_2 containing the origin, both with $K = 0$, both with the tangent plane at the origin being the xy-plane, and such that near the origin S_1 lies all to one side of the xy-plane while S_2 lies on both sides of the xy-plane.