PROBLEMS

0: Show that the differential of the Gauss map is self-adjoint, i.e.,

$$dN_p(v).w = dN_p(w).v$$

for all $v, w \in T_pS$ and all $p \in S$. This exercise is important!

1: Show that the cone $\{(x,y,z) : z = \sqrt{x^2+y^2}\}$ is not a surface

2: Show that

- If $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ is smooth then its graph is a surface
$$graph(f) = \{(u,v,f(u,v)) : (u,v) \in U\};$$
- If $f : \mathbb{R}^3 \to \mathbb{R}$ a smooth function such that $t \in \mathbb{R}$ is a regular value, i.e., $\nabla f(p) \neq 0$ for all $p \in f^{-1}(t)$, then $S = f^{-1}(t)$ is a surface (called level surface).

3: Show that if the surface S is contained in $\cup_{i\in\mathbb{N}}\mathbf{x}_i(U_i)$, where each $(U_i, \mathbf{x}_i)$ is a chart then a function $f : S \to \mathbb{R}$ is differentiable iff $f \circ \mathbf{x}_i : U_i \to \mathbb{R}$ is differentiable for all $i \in \mathbb{N}$.

4: S is a surface. Given a chart $(U, \mathbf{x})$ where $\mathbf{x}(q) = p$,, show that $T_pS = d\mathbf{x}_q(\mathbb{R}^2)$.

5: Given a surface S and $p \in S$, show the existence of a neighborhood $V \subset \mathbb{R}^3$ of p so that $V \cap S$ is the graph of a function defined over T_pS, more precisely, there is $U \subset T_pS$ an open set containing the origin and $f : U \to \mathbb{R}$ a differentiable function so that

$$V \cap S = \{q + f(q)n : q \in U\}$$

where n is a unit vector perpendicular to T_pS.

6: Consider the surface $S = f^{-1}(t)$ for some $f : \mathbb{R}^3 \to \mathbb{R}$, where t is regular value. Show that

$$T_pS = \{w \in \mathbb{R}^3 : w.\nabla f(p) = 0\}.$$

and conclude that $N(p) = \frac{\nabla f(p)}{|\nabla f(p)|}$.

7: Let S be an orientable surface. Show that the Gauss map is differentiable.