An introduction to Lie groups, symmetries, and symplectic geometry
Problem Set 1

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$M_n(\mathbb{R})$ denotes the space of $n \times n$ matrices with real entries. Denote by $I_n \in M_n(\mathbb{R})$ the identity matrix. The n is ommited from the notation when it is clear from context.

The *orthogonal group* is

$$O(n) = \left\{ A \in M_n(\mathbb{R}) | {}^t A A = I \right\}.$$

The *special linear group* is

$$SL(n, \mathbb{R}) = \{ A \in M_n(\mathbb{R}) | \det(A) = 1 \}.$$

Denote $J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \in M_{2n}(\mathbb{R})$.

The *(real) symplectic group* is

$$\mathrm{Sp}(n, \mathbb{R}) = \left\{ A \in M_{2n}(\mathbb{R}) | {}^t A J_n A = J_n \right\}.$$

Exercise 1. *Find the critical points and critical values of the function*

$$\begin{aligned} \mathbb{R}^2 &\to \mathbb{R}^1 \\ (x, y) &\mapsto x^2 - y^2 + (x^2 + y^2)^2. \end{aligned}$$

Exercise 2. *Show that* $SL(n, \mathbb{R})$ *and* $\mathrm{Sp}(n, \mathbb{R})$ *are submanifolds of* $M_n(\mathbb{R})$ *and* $M_{2n}(\mathbb{R})$*, respectively, and compute their dimensions.*

Exercise 3. *Compute* $T_I G$ *for* $G = O(n), SL(n, \mathbb{R}), \mathrm{Sp}(n, \mathbb{R})$ *and show that*

$$T_a G = \{ av | v \in T_I G \}.$$

Hint: *Consider* $L_a : G \to G$ *where* $L_a(g) = ag$.