This inversion switches the interior and exterior of the circle and preserves the circle itself.
It is like a "reflection on a curved mirror", it reflects in a distorted way.

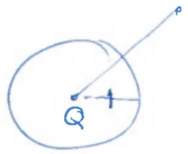
Let $S^2 = \text{plane} \cup \infty$

We can think of the inversion as a map $S^2 \longrightarrow S^2$ taking p to ∞ and vice-versa.

Another way to do this is to think of the Riemann sphere $\mathbb{C} \cup \infty = \hat{\mathbb{C}}$

Many interesting maps on $\mathbb{C}$ extend to $\hat{\mathbb{C}}$ and it makes sense to talk about holomorphic maps $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$, they are the meromorphic maps.

Suppose C is a unit circle $\subset \mathbb{C}$ centred at Q



Inversion: $z \longmapsto \frac{1}{\bar{z}}$

Exercise: This map takes round circles to round circles. Not quite! The exercise is to formulate a correct version of this statement and prove it.

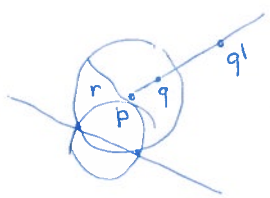
Tomorrow: will classify invertible maps $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ taking round circles to round circles
will need to see what this means

We'll see this is a 6-dimensional group that leads to hyperbolic geometry.

Note: A pole of $f(z)$ at ∞ of order k means that $f(\frac{1}{z})$ has a pole of order k at 0.

Lecture 3.

Inversions: fix C with center p. For any other q



$pq \parallel pq'$
$pq \cdot pq' = r^2$

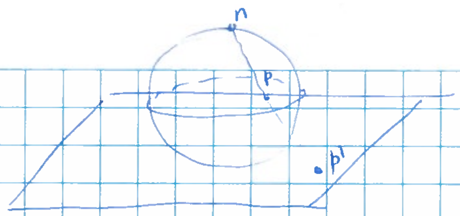
takes round circles to round circles. Not quite
a straight line through 2 pts in the circle will be taken to
a circle through the center and vice-versa
a straight line through the origin will be sent to a straight line of the same kind.

$S^2 = \mathbb{C} \cup \infty$
$S^2 = \{(x,y,z) : x^2+y^2+z^2 = 1\}$ is a Riemannian manifold $\Rightarrow$ nice complete metric space of diameter π

What is a round circle on S^2?
(i) a set of points equidistant to some $p \in S^2$
(ii) intersection of S^2 with a plane in $\mathbb{R}^3$

These are 2 different ideas of what a circle. Luckily they coincide. One way to see this is to note they are the orbits of one parameter groups of isometries namely rotations around an axis.

There is a std method to go between the sphere and the plane, namely <u>stereographic projection</u>



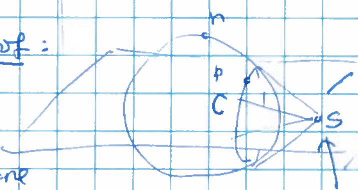
$p \mapsto p'$

$n \mapsto \infty$

<u>claim</u>: stereographic projection takes round circles on S^2 to round circles and straight lines on $\mathbb{R}^2$ and conversely.

The equations defining circle and lines in the plane correspond to slightly different equations defining circles on S^2 and these can be used to write equations for circle packings on the sphere.

<u>Proof</u>:



osculating cone to the circle

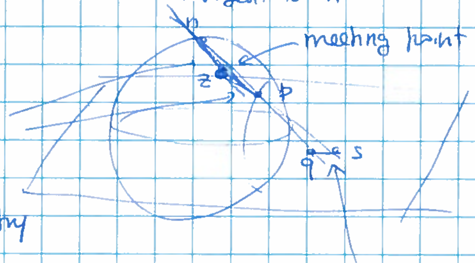
at each point the line leaving the point in the circle is tangent to the sphere and $\perp$ to the circle.

plane containing the apex of the cone

apex of the cone

tangent to n

meeting point of two lines (tangent at n and generating line of the cone through p



These two lengths are equal by symmetry

qs parallel to nz

pnz and pqs are similar triangles

It now follows that pq and qs are equal in length.

It follows that the locus of the qs is a circle centred at s. $\square$

This is one approach. One can also prove this identity with brute force using equations (which by a theorem of Tarski can even be ~~done~~ solved algorithmically).

To prove that inversions take round circles to round circles - It's enough to do it for one specific circle because there are similarities, taking any circle to any circle and so $z \mapsto az+b$

~~therefore~~ an inversion on a different circle is conjugate via a similarity.

(in the image of the stereographic projection)

Consider the unit circle^{Y}. Then the stereographic projection will take inversion in that circle into reflection on the plane of the equator on the sphere and therefore takes round circles to round circles.

This is a useful method in geometry. When there is an action of a group of symmetries, can use it to reduce the problem to particularly nice, symmetric cases.

$\alpha : S^2 \longrightarrow S^2$ taking round circles to round circles invertibly. (+ orientation preserving)
$\quad \| \qquad \|$
$\quad \hat{\mathbb{C}} \qquad \hat{\mathbb{C}}$

The proof we saw last time (taking limits of small circles) shows α is conformal and hence is holomorphic (as a map from a Riemann surface to itself)

Every holomorphic map $\hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$ is a rational map

$$z \longmapsto \frac{p(z)}{q(z)} \quad \text{for some polynomials } p, q$$

$\deg = \max(\deg p, \deg q)$ must be 1 so $\deg p, \deg q \leqslant 1$

and $\alpha(z) = \dfrac{az+b}{cz+d}$ $a, b, c, d \in \mathbb{C}$ (this will be invertible as long as $ad-bc \neq 0$)

If we scale a, b, c, d by a complex number α does not change so we may as well assume $ad-bc=1$.

2×2 matrices with $\det = 1$ is denoted by $SL(2;\mathbb{C})$.

This group acts on $\hat{\mathbb{C}}$

± 1 acts trivially so this is actually an action of $PSL(2;\mathbb{C}) = SL(2;\mathbb{C})/\pm 1$

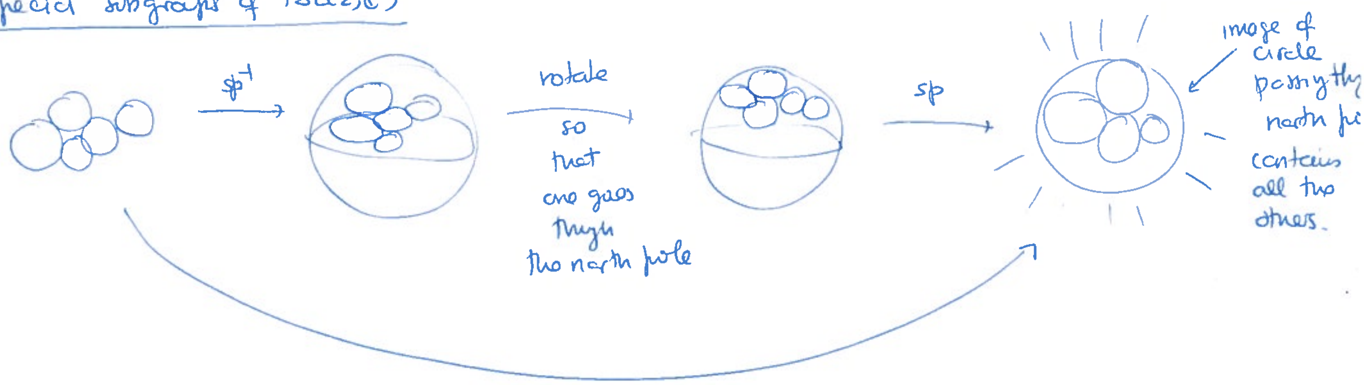
This is exactly the group of orientation preserving invertible maps taking round circles to round circles. and preserving the orientation.

$PSL(2;\mathbb{C})$ has 3 complex dimensions so 6 real dimensions. These are the 6 dimensions of global symmetries. One can see this 6 degrees of freedom as follows

3 come from rotations (around the North, East and front "poles")

3 come from dilations relative to those poles.

Some special subgroups of $PSL(2;\mathbb{C})$



Consider the unit disk $\mathbb{D} \subset \mathbb{C}$.

The subgroup of $PSL(2;\mathbb{C})$ taking $\mathbb{D}$ to itself is $\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$ so that $|\alpha|^2 - |\beta|^2 = 1$

Check and also that it takes the inside of unit circle to itself

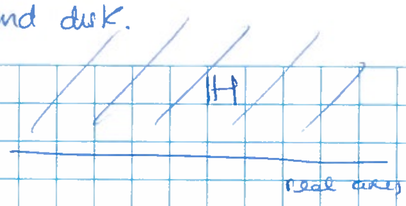
This has real dimension 3

It is a nice group of symmetries called $PSU(1;1)$

TÉCNICO LISBOA

Note that it contains rotations of the round disk.

$\mathbb{H} = \{ z : \operatorname{Im} z > 0 \}$

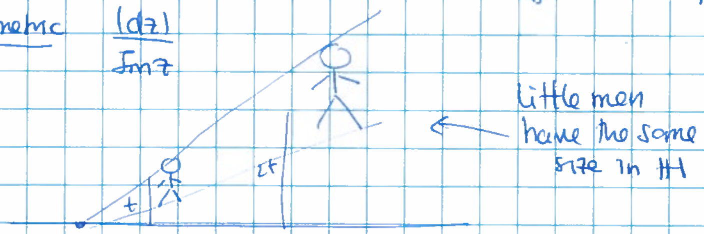


Subgroup of $PSL(2;\mathbb{C})$ taking $\mathbb{H}$ to itself. Must take real axis to itself.
One way to do this is to take real matrices and
Indeed the group is

$$PSL(2;\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \; a,b,c,d \in \mathbb{R} \right\} \Big/ \pm I$$

There is an inversion taking the unit circle to $\mathbb{H}$ so by the previous calculation the group preserving $\mathbb{H}$ must also be 3 dimensional. A dimension count (using the fact these are Lie groups) then shows that the real matrices must be the only elements fixing $\mathbb{H}$.

Define a metric on $\mathbb{H}$ by taking the euclidean metric $|dz|$ and scaling it at every point by $\frac{1}{\operatorname{Im} z}$ to get the hyperbolic metric $\frac{|dz|}{\operatorname{Im} z}$



dilation through a point in the boundary therefore preserve distances as do translations

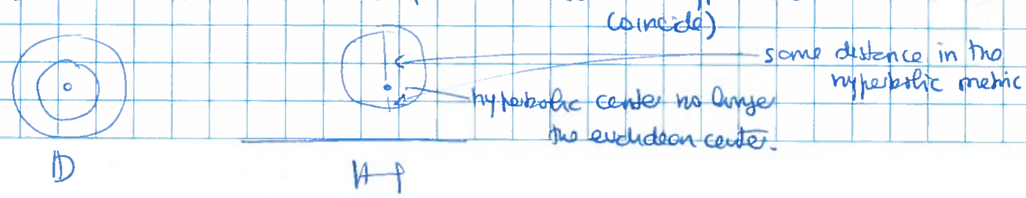
To see that the whole $PSL(2;\mathbb{R})$ preserves the metric we need to do some calculations.

There is a conformal map $\mathbb{H} \longrightarrow \mathbb{D}$ $\Rightarrow$ angles in $\mathbb{D}$ ~~are the same as~~ under the (pushed forward) hyperbolic metric ~~is the same as~~ are the same as in the Euclidean plane so again the pushed forward metric must be a scale multiple of the euclidean metric and indeed it is $\frac{2|dz|}{1-|z|^2}$

~~One can see that PSL(2;R) preserves these so the group that preserves the~~

From this one can see that $PSL(2;\mathbb{R})$ preserves the hyperbolic metric which can also be checked directly.

Hyperbolic circles are the same as euclidean circles: to see this move the circle to the disk and ~~make~~ move the center to the center of the disk (so that hyperbolic and euclidean circles coincide)



lecture 4

Remark: We proved that a smooth orientation ~~preserving angle~~ preserving map taking round circles to round circles is ~~[illegible]~~ a similarity. It can be shown that continuity is enough. One may ask whether even continuity is necessary and in fact, surprisingly there are discontinuous maps with this property.

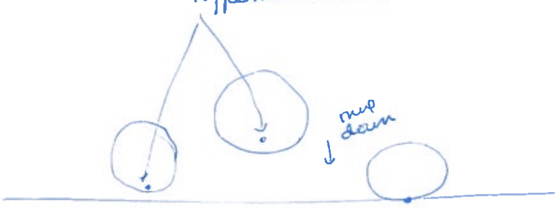
$(z-c)\overline{(z-c)} = r^2$ $\qquad \sigma:\mathbb{C}\to\mathbb{C}$ ~~crazy~~ extension of a "crazy" Galois automorphism $\mathbb{R}\xrightarrow{\sigma}\mathbb{R}$ (these are not even measurable)

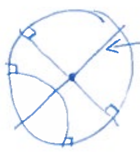
Since σ commutes with complex conjugation the image of the circle is

$$(w-\sigma(c))(w-\overline{\sigma(c)}) = \sigma(r)^2$$ which is another circle

Geometry of the hyperbolic plane:

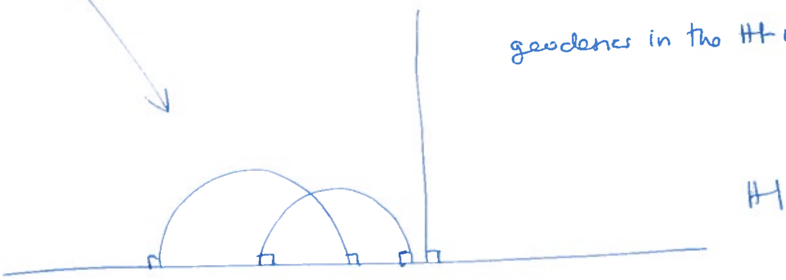
~~$\cos\alpha = \sin\beta\sin\gamma\cosh a - \cos\beta\cos\gamma$~~





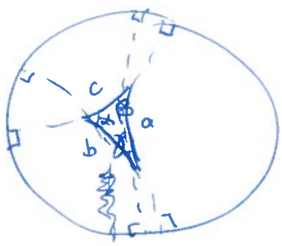
straight lines through the center are geodesics (as they are the fixed points of an isometry)

arcs of round circles perpendicular to the boundary are also geodesics (??) and these must be all by seeing how many there are.



geodesics in the H model

to find length in the hyperbolic plane just integrate arc length along geodesics

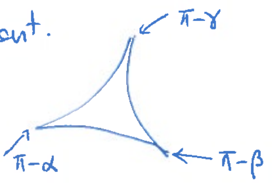


interesting formula (exercise): $\cos\alpha = \sin(\beta)\sin(\gamma)\cosh(a) - \cos(\beta)\cos(\gamma)$

note that as the triangle shrinks $\cosh a \to 1$ and this approaches the ~~[illegible]~~ euclidean formula.

$\alpha = \pi - \beta - \gamma$.

One can check this equation can be solved iff $\alpha+\beta+\gamma<\pi$ and that all hyperbolic triangles with same angles are congruent.



The curvature of the hyperbolic metric is exactly -1 so Gauss-Bonnet from Andre's lecture gives

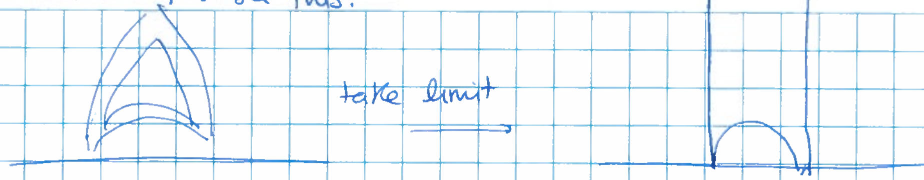
geodesic curvature, concentrated at the vertices

$$-\int_\Delta K = 2\pi\chi(\Delta) - \int_{\partial\Delta} k_g$$

$$\Leftrightarrow \quad \text{area}(\Delta) = 2\pi \cdot 1 - (\alpha+\beta+\gamma-3\pi)$$

$$\Leftrightarrow \quad \text{area}(\Delta) = \pi - (\alpha+\beta+\gamma)$$

Here's another way to see this:



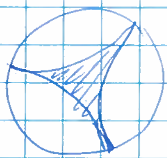
ideal triangle with a vertex at ∞

This is a way of compactifying the space of hyperbolic triangles

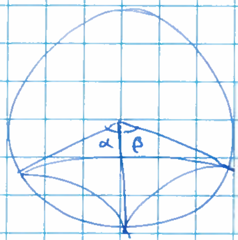
$B(\alpha)$ = area of this triangle with 2 vertices in the boundary

← semi-ideal triangle

unique up to isometry triangle with 3 vertices at ∞

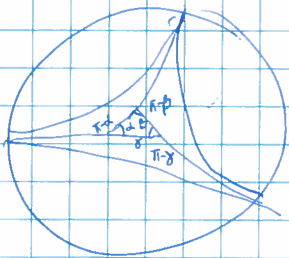


area $= \pi$ by convention (this is the convention that gives curvature -1)



$\Longrightarrow \quad B(\alpha) + B(\beta) = B(\alpha+\beta) + \pi$

There is only one continuous solution namely $\boxed{B(\alpha) = \pi - \alpha}$



This picture shows that

$$A(\alpha,\beta,\gamma) + B(\pi-\alpha) + B(\pi-\beta) + B(\pi-\gamma) = \pi$$

and gives a beautiful synthetic proof of Gauss-Bonnet for triangles. One can prove Gauss-Bonnet formula from this by ~~taking this formula~~ using this formula for small triangulations and carefully estimate the error terms.

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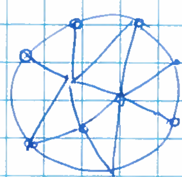
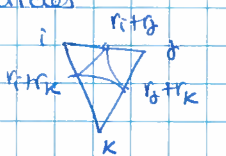
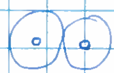
circle packing ⇝ triangulation of the disk



suppose we know not the centers but the radii of the circles



$r_i, r_j, r_k \longrightarrow$ shape of triangle $\rightarrow$ angles of triangle.



put the center of the first two somewhere, the others are inductively determined by intersecting circles of given radius the

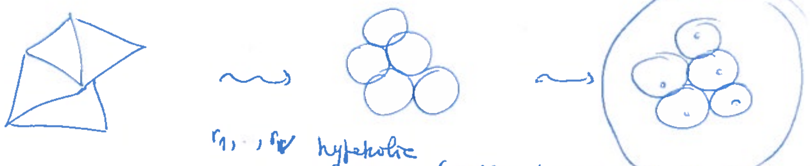
So collection of lengths coming from a circle packing ⇝ construction of the packing.

Now a collection of numbers $\{r_1,\dots,r_V\}$ determines a "immersed" circle packing"

meaning tangencies are correct but there may be self intersections.

$(\Leftrightarrow)$ $\sum_{\text{vertex}}$ angles $=2\pi$ (at internal vertices)

triangulation $\to$ circle packing (Can draw a big ~~circle~~ circle around them and think of them in the hyperbolic plane.

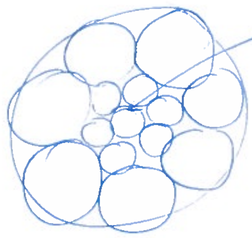


∘ = hyperbolic centers

$r_1,\dots,r_V$ hyperbolic lengths (different collection for the same picture)

$\to \{\alpha_1,\dots,\alpha_e\}$. If $\sum_{\{\text{internal vertex}\}}$ angles $=2\pi$ then we get (immersed) circle packing on the hyperbolic plane.

Would like a circle packing in hyp. plane and would like it to be as big as possible. By Gauss Bonnet the area is determined by the outside angles (of the outer circles) and this is biggest when the angles are 0 ie. the outside circles are horocircles.



greatest enclosed area

Let $\Phi=\{(r_1,\dots,r_V):\sum$ angles at every internal vertex $\geq 2\pi\}$ (this is weaker).

$>2\pi$ ⇝ atom of negative curvature contributing to integral but not the area

If we look for maximal area the ~~area~~ angles should be 2π at every vertex.

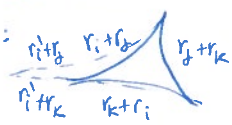
Look for $\theta\in\Phi$ for which area $=\sum$ areas of triangles is as big as possible.

~~Next the circles~~
Then the angles at the boundary will all be 0 and will have a picture as above.

~~Absolutely~~ Φ is a subset of a product of intervals and the inequality is not strict, so Φ is compact so the maximum must exist as long as $\Phi\neq\emptyset$ (as area is continuous).

Assume $\Phi\neq\emptyset$.

<u>Lemma</u>:



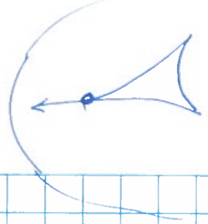
suppose we increase r_i a little bit. Then the triangle will bow out a little bit.

2 angles at j,k increased and the area too so the angle at i must ~~do~~ have decreased.

Now if at our maximum the sum of angles at an internal angle is $>2\pi$. Can lower it but keep it above 2π in such

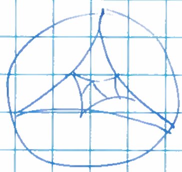
a way that ~~all~~ at all the other cycles they increase
But this contradicts the fact that the area is a maximum.

If a boundary vertex is not ∞
can move it to the boundary
thus increasing the area



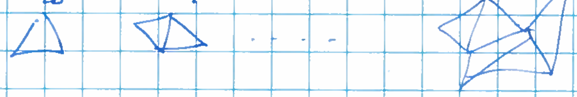
□

~~Abstract~~ Triangulation of S^2 = triangulation of D^2 with 3 boundary circles



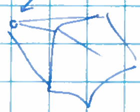
removing sides can get a sequence of smaller and smaller triangulations

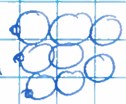
T_0 T_1 . . . . $T_{|T|}$



Inductive construction. Given a circle packing for T_i construct one for T_{i+1}

new vertex





need a circle tangent to those points

must deform the previous packing.

make the packing T_i optimal as before

all the points we want ~~are~~ tangent to the inner circle but maybe there are more

Now can remove the extra tangencies (??)

This can easily be turned into an algorithm to find the packing.