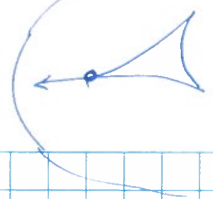
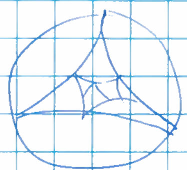
a way that ~~all~~ at all the other angles they increase.
But this contradicts the fact that the area is a maximum.

If a boundary vertex is not ∞
can move it to the boundary
thus increasing the area

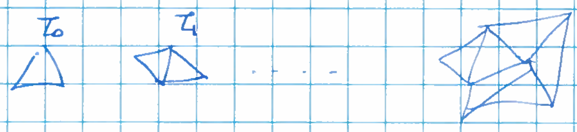


$\square$

~~Alsatie~~ Triangulation of S^2 = triangulation of D^2 with 3 boundary circles

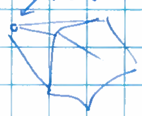


removing sides can get a sequence of smaller and smaller triangulations

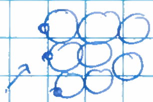


T_0 T_1 . . . . $T_{|T|}$

Inductive construction. Given a circle packing for T_i construct one for T_{i+1}



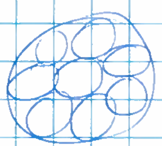
new vertex



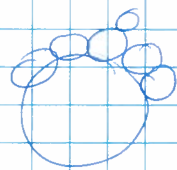
need a circle tangent to those points

must deform the previous packing.

make the packing T_i optimal as before



do an inversion



all the points we want tangent to the inner circle but make them one more

Now can remove the extra tangencies (??)

This can easily be turned into an algorithm to find the packing.

Lecture 5

Last time we saw an existence proof for a circle packing associated to a triangulation of the sphere. Moreover it was a constructive process which converges very quickly in practice.

Today: applications of circle packings to other parts of mathematics, in particular 3d hyperbolic geometry including the ~~great~~ first step in the proof of geometrization of Haken 3d manifolds.

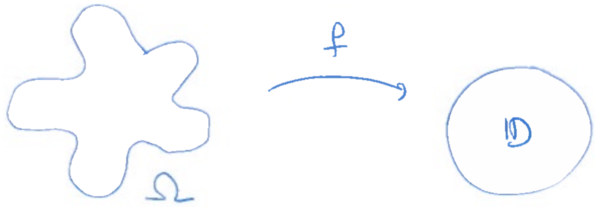
3d manifold is Haken if it can be decomposed along a finite number of incompressible ~~manifolds~~ surfaces into balls. (which will have a certain combinatorial structure). Finding a hyperbolic model for the case of balls involves finding a hyperbolic polytope with dihedral angles $\pi/2$ and this is where the circle packings come in

Apps:

0) (Silly)

planar graph can be rectified: ~~generate found/there~~ make it part of a triangulation and find a circle packing ~~set~~ for it.

① Riemann mapping theorem



If we normalise: pick a point going to the centre and a direction say to the real axis. Then there is a unique biholomorphic map (this fixes 3 degrees of freedom which cancel out the symmetries of $\mathbb{D}$)

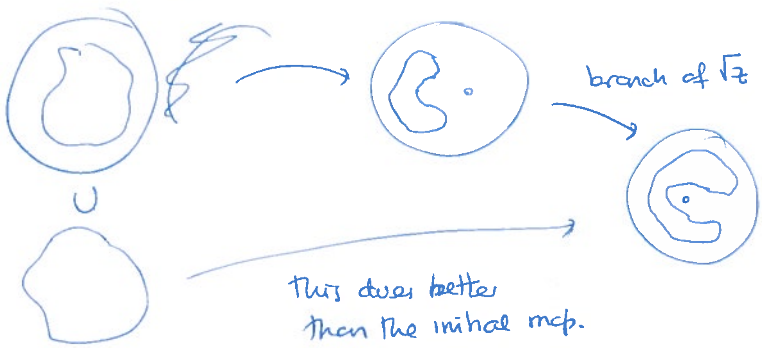
<u>typical proof</u>: Use a kind of maximum principle. If we move $\mathbb{D}$ around itself taking the centre to itself must have $|f'(0)|<1$ and its only 1 if it is an isometry (Schwarz lemma).

This implies that there is a canonical hyperbolic metric on any domain. If we have a domain contained in another

then the inclusion will strictly shrink distances. This is a restatement of the Schwarz lemma in terms of hyperbolic metrics.

Idea look at all injective holomorphic maps from Ω to D sending a particular point to the origin and ask for the one with biggest derivative. There exists such a function. We can control the size of the derivatives using Cauchy integral formula $\Rightarrow$ the space of such maps is pre-compact.

The claim is that this map has to be surjective. Suppose not. By a Möbius transformation move it so that the center is missed.



(z^2 shrinks distances $\Rightarrow \sqrt{z}$ increases)

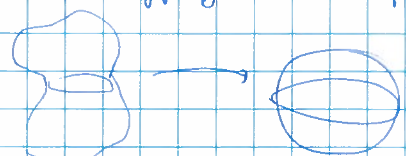
This proof is very nice but it's not very constructive. One could implement this as an algorithm but it probably converges very slowly. Here's something more constructive.

take the obvious map to the disk.
It takes some circles to circles.
to get a better map fill the hollow region with smaller circles. ~~[illegible]~~ Continue like this. In the limit infinitesimal [illegible] will [illegible] to infinitesimal circles so this will be

The existence of holomorphic. Such a proof was conjectured by Thurston in the Bieberbach conference and later proved by Sullivan + - - - .

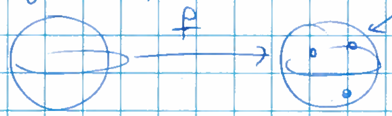
This actually converges pretty fast to a uniformization of a domain

Another version of Riemann mapping theorem says if we have a two sphere there is a conformal map to the round sphere



(ie preserves infinitesimal circles)

Uniformizing this surface leads to a rational map



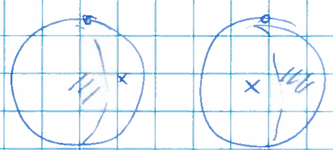
$f(z) = \frac{p(z)}{q(z)}$ rational map $\deg = d$

This looks like a d-sheeted copy $z \mapsto z^d$ around each of the critical points (d varying with the critical point).

Can now consider the ~~Riemann~~ space associated to this branched cover. Angles still make sense

If we branch like $z \mapsto z^{35}$ divide angles by 35 ...

Example:



At the N and S poles the total angle is 4π.

Would like to find a conformal map from this to the round sphere

Take a circle packing of the 2 spheres $\rightsquigarrow$ get a circle packing on the spliced surface.

If each circle had a certain number of circles around it the same in the upstairs except for the north + south pole where the # circles gets doubled.

So we have a circle packing of the round sphere into the spliced surface and composing it with the rational map we get a combinatorial model of the map which can be conveniently implemented on the computer.

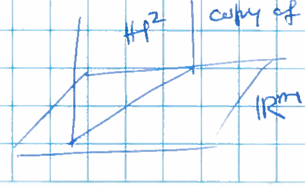
It is convenient that the critical values of the 2-sphere are centers of the circles. Since the circle packings are discrete this is not ~~at~~ always possible. In this case can use "packed packings" Everything then works as before (this has been done by Laurent Bartholdi).

This now allows one to investigate dynamic maps of degree 30 when before the highest possible was about 5.

(totally geodesic) copy of $\mathbb{H}^2$ for any line on $\mathbb{R}^{n-1}$

② Hyperbolic geometry in dimension 3

$\mathbb{H}^n = \{ (x_1, .., x_n) \in \mathbb{R}^n : x_n > 0 \}$



$\partial \mathbb{H}^3 = \{ (x_1, x_2, 0) \}$ so the Riemann sphere appears as
$\underbrace{\quad}_{x_1 + i x_2}$

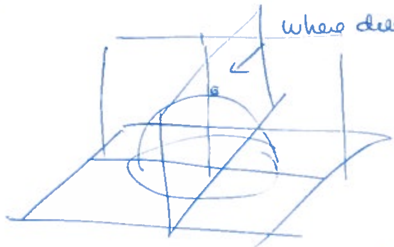
the boundary of $\mathbb{H}^3$ and in fact the conformal maps on bdry extend to isometries of $\mathbb{H}$ and these are all the isometries. i.e. the isometry group is $PSL(2, \mathbb{C})$.

Geodesics = geodesics contained on a totally geodesic $\mathbb{H}^2$.

perpendicular spheres are again totally geodesic $\mathbb{H}^2 \subset \mathbb{H}^3$

How do isometries act?

where does the point go?



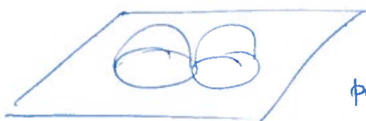
point is determined by the 3 round circles in the boundary.

the isometry on the boundary will send the 3 round circles to three other and the point will then map to the intersection of the corresponding $\mathbb{H}^2$s

image point.

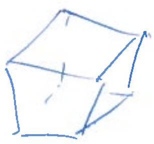
<u>Note</u>: This goes on for any dimension: there is a strong correspondence between hyperbolic geometry of $(n+1)$-manifolds and conformal geometry of n-spheres.

$$\left\{ \begin{array}{c} \text{configuration of} \\ \text{round circles} \\ \text{in } S^2 \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{configuration of} \\ \text{totally geodesic} \\ \mathbb{H}^2 \subset \mathbb{H}^3 \end{array} \right\}$$



pair of $\mathbb{H}^2$'s which are assymptotically circles

Polyhedra in hyperbolic space = regions limited by totally geodesic $\mathbb{H}^2$'s



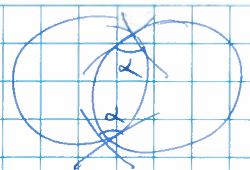
Convex object in $\mathbb{R}^3$. Can look at half spaces that don't contain. They cut out the polyhedra

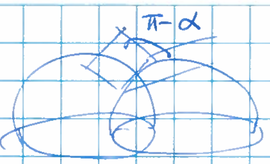
Convex polyhedra = finite intersection of half spaces

The same's true in the hyperbolic case.

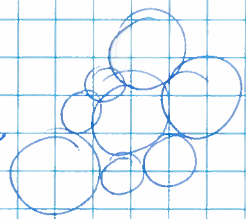
So suitable configurations of circles in the plane correspond to hyperbolic polyhedra.

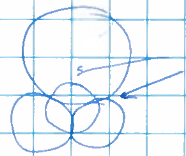
Correspondence between intersection of bdry circles and dihedral angles:



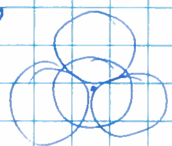


So if we want to find polyhedra with dihedral angles π/2 is the same as finding a configuration of circles with angles of intersection π/2.

C 

Lemma:  angles are all π/2

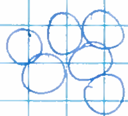
Proof: Apply a conformal transformation to make the circles symmetric around the origin

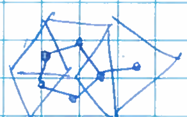


then the angles are π/2 (??) □

Do as in the lemma for the configuration above and get a polyhedron with dihedral angles π/2

C' = dual triangulation





triangulation

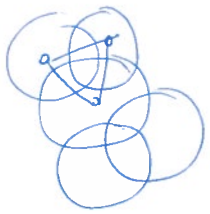
... ?

If we have a polyhedron with certain properties (??) we can get a unique convex hyperbolic polyhedron with dihedral angles π/2.

interesting because the reflections on the faces commute ⇝ tesselation of $\mathbb{H}^3$ w/ certain grp of symmetries abstractly generated by reflections in the circles (generalised Coxeter group)

Quotient is hyperbolic space with finite volume. The group may have torsion but there's always a finite index subgroup and the quotient by this leads to a hyperbolic 3 manifold which are therefore easy to produce from circle packings.

Not all can be obtained in this way. Can also consider circle patterns



If the nerve of the cover is a triangulation of the 2-sphere we only need to know the ~~~~ radii

This will give a more complicated formula for the length of the edges in terms of the radii but other than this we have the same setup as before.

Kirby - Andreev - Thurston: trivalent graphs in 2 sphere can be realized as the edges of a hyperbolic polyhedron with dihedral angles $\pi/2$ ~~unless some simple~~ as long as certain (?) simple combinatorial conditions are satisfied.

This is the first step in a famous theorem of Thurston: Geometrization of Haken 3 manifolds