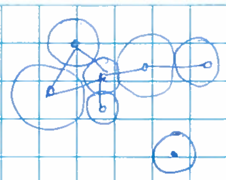
TÉCNICO LISBOA Danny Calegari - Hyperbolic Geometry

Will start with a simple puzzle in Euclidean geometry and see that lurking behind it lies Hyperbolic Geometry.

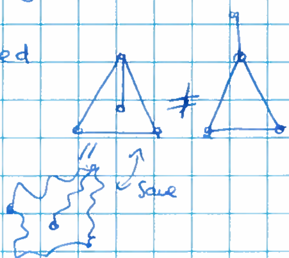
A circle packing is a collection of circles in the plane with disjoint interiors. They're allowed to touch along the boundary



To such an object we can associate a tangency graph
vertices $\longleftrightarrow$ circles
edges $\leftrightarrow$ ~~circles are tangent~~ tangencies

Remarks: (i) this graph has no self loops ✗
(ii) there are no multi-edges ✗
(iii) planar (ie can be embedded in the plane - it is by definition embedded in the plane)
For us actually a planar graph is not just a graph that can be embedded in the plane but an embedding up to isotopy

Example: 2 different plane embed



Motivating Question: Given a planar graph without self-loops multi-edges is there a circle packing whose tangency graph it is? We'll see the answer yes.

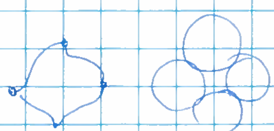
This is a topological problem but we'll see that if we make it geometric (rigid) then there is a unique solution, up to a certain global group of symmetries which is the same for all circle packings.

It's easier to tell whether two geometric objects are the same because the relevant equivalence relation is much simpler.

This is the prototype of certain arguments which are fundamental in 3-manifold topology.

This is the first step involved in the proof of Thurston's geometrization theorem

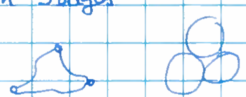
Back to our problem: For arbitrary Γ the situation is under-determined (the solution is far from unique). e.g.



can move continuously

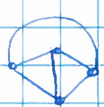
This movement can be achieved with rigid coins

This lack of uniqueness is intrinsic to this problem.
If we had a cycle with 3 edges

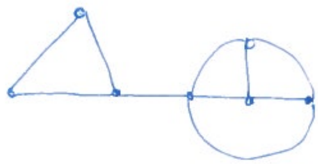


then there is a symmetry of the entire plane that moves the circles but not something like before.

Definition: A triangulation Δ is a planar graph which is connected and such that all complementary regions are 3-sided.

Eg  is an example. We will in fact think of this as a triangulation of the sphere - the one point compactification of the plane.

Γ

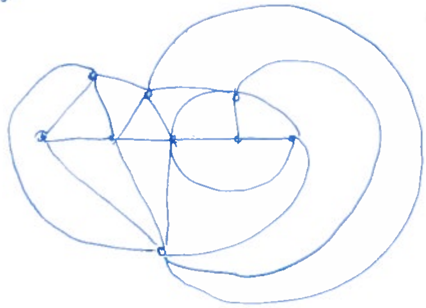


would like to embed this in Γ' as a full subgraph so that Γ' is a triangulation.

means that given 2 vertices in Γ' which were already in Γ there will be no new edges between them

the complementary region is an octagon.

Γ'



this is a triangulation, corresponding to ~~[illegible]~~ cutting up a sphere into several triangles.

If we have a circle packing for the triangulation we also obtain one for Γ^*, the original graph.

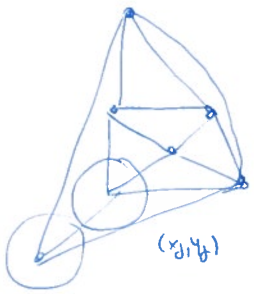
Thus we can add an extra constraint to the graph - being a triangulation. This is locally rigid. In fact we'll see that ~~[illegible]~~ in this case the circle packing will be unique up to a symmetry of the sphere taking circles to circles.

We've added more constraints so that the solution is unique, which makes it easier to find. This is something that's often done even in engineering - add ~~[illegible]~~ artificial constraints so as to break symmetry and ensure that iterative methods converge.

How to circle pack?

Naive idea:
- formulate a circle packing as a solution to a system of equations
- solve equations

Note these are independent problems.



Coordinates of the center — radius

each vertex $i \rightsquigarrow (x_i, y_i, r_i)$

for each edge $ij \rightsquigarrow$ equation $(r_i+r_j)^2 = (x_j-x_i)^2 + (y_j-y_i)^2$

How plausible is it that there is a solution? that it is unique?

topology: Euler's formula Suppose we have a "cellulation" of S^2 (i.e. a graph so that the components of the complement are $\approx D^2$). Then $|F|-|E|+|V| = 2$

(the answer would be difference if the surface was not sphere). This combinatorial formula actually carries a lot of information about solutions to certain equations.

For a triangulation, each face has 3 edges and each edge has 2 faces so $3|F| = 2|E| \iff |F| = \frac{2}{3}E$

Hence $|V| = \frac{1}{3}|E| + 2 \iff 3|V| = |E| + 6$ ← accounts for the missing group of symmetries.

($3|V|$: number of variables; $|E|$: number of equations)

In order to get rid of the symmetry can pick 3 vertices and decide where to put them. Then ~~those~~ we would expect there to be only one solution.

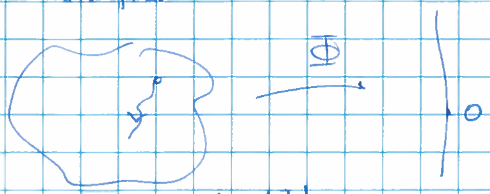
Pick the 3 outside vertices and send them to a specific equilateral triangle.

(Important problem with this approach: there could be overlaps or extra tangencies.)

Nevertheless how would we solve these equations?

$$\Phi = \sum_{edges} \Big(\underbrace{(r_i+r_j)^2 - (x_j-x_i)^2 - (y_j-y_i)^2}_{\text{measures failure of the equation to be satisfied}} \Big)^2$$

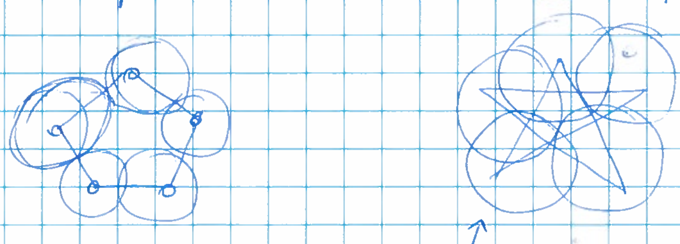
Solutions correspond to level set $=0$ of this function.



space of possibilities of dim $3|V|-6$

Start at any point and flow in the direction of $-\nabla\Phi$. Computationally this is quite easy to implement as Φ is a simple polynomial.

<u>Note</u>: ~~we~~ in these equations we did not take into account the fact that it is a planar graph, or



different configuration with overlaps

Maybe these do correspond under a map from the sphere to the sphere, but not invertible. We'll need to approach this in a more intelligent way next time.

In other surfaces we can do the same thing. On a torus $|V|-|E|+|F|=0$ so there are 6 less parameters. Since the symmetry of the torus is 2 dimensional we expect there to be no solutions, which is indeed the case.