TÉCNICO LISBOA

This tells us that matrix groups have very special tangent spaces. One of them determines the others
In fact we'll see they tell us a lot about the group

LECTURE 2 : Today need to make a connection between differential equations and Lie groups

$G \subseteq GL(n;\mathbb{R}) \subset M_n(\mathbb{R})$ (invertible $n\times n$ matrices) is a group and a smooth manifold.

$\mathfrak{g} = T_I G \Rightarrow T_a G = \{ av \mid v \in \mathfrak{g} \} =: a\mathfrak{g}$

This leads us to an important construction: Left invariant vector fields

$v \in M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ $\qquad X_v(a) = av$ is a vector field on $M_n(\mathbb{R})$

Note that $X_v$ is tangent to $G$ at all points of $G$, if $v \in \mathfrak{g}$.
Recall: A curve $\gamma : \mathbb{R} \to M_n(\mathbb{R})$ is an integral curve of a vector field $X$ if $\gamma'(t) = X(\gamma(t))$ for all $t \in \mathbb{R}$.

Not all vector fields have global integral curves but the existence and uniqueness theorems for ODEs guarantee that there is a unique solution near a given point (i.e. defined in a small interval)

Prop: The integral curves of $X_v$ are $\gamma(t) = \gamma(0)e^{tv}$ where $e^v = \underbrace{I + v + \frac{1}{2}v^2 + \frac{1}{6}v^3 + \cdots}$

converges absolutely and uniformly on compact sets just like the usual exponential
[for any choice of norm on matrices - they are all equivalent]

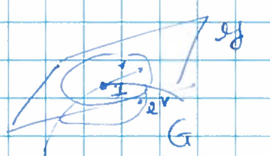
Note that if we have an integral curve through $I$ and we multiply on the left by $a$ we get an integral curve through $a$.

Proof: $\gamma'(t) = \gamma(0) v e^{tv} = \gamma(0) e^{tv} v = \gamma(t) v = X_v(\gamma(t))$ $\square$
differentiate term by term

Corollary: $e^v \in G$ for all $v \in \mathfrak{g}$

Prop: If $\exp : \mathfrak{gl}(n;\mathbb{R}) \underset{= T_I GL(n;\mathbb{R}) = M_n(\mathbb{R})}{} \to GL(n;\mathbb{R})$ is $\exp(v) = e^v$. Then $\exp'(0_n) : \mathfrak{gl}(n;\mathbb{R}) \to \mathfrak{gl}(n;\mathbb{R})$ is the identity.

Corollary: $\exp : \mathfrak{g} \to G$ is locally a diffeomorphism




In particular there is an open neighborhood $U$ in $G$ of the identity so that $\log : U \to \mathfrak{g}$ exists and is smooth where

$$\log(e^x) = x \quad \text{for all } x \text{ with } |x| < \varepsilon$$

$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots$ will converge absolutely as long as all the eigenvalues of $x$ have absolute value less than 1.

Example: $N_3 = \left\{ \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} : a,b,c \in \mathbb{R} \right\} \subset GL_3(\mathbb{R})$

$\mathfrak{n}_3 = \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} : x,y,z \in \mathbb{R} \right\}$

It's easy to check that $\exp \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & x & z+\frac{xy}{2} \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$. Here $\log \mathfrak{n}_3 \to N_3$ exists globally

Example: $SL(2;\mathbb{R}) = \{ A \in M_2(\mathbb{R}) \mid \det A = 1 \}$

Every matrix can be factored as a positive definite symmetric matrix of det 1 and an orthogonal matrix:

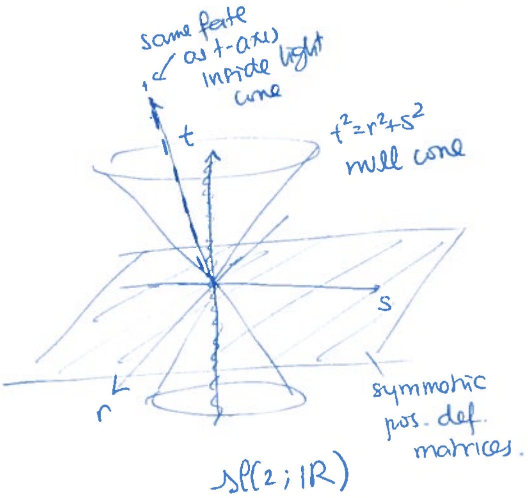
$$A = \begin{pmatrix} \sqrt{1+a^2+b^2}+a & b \\ b & \sqrt{1+a^2+b^2}-a \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad \begin{matrix} a,b \in \mathbb{R} \\ \theta \in S^1 \end{matrix}$$

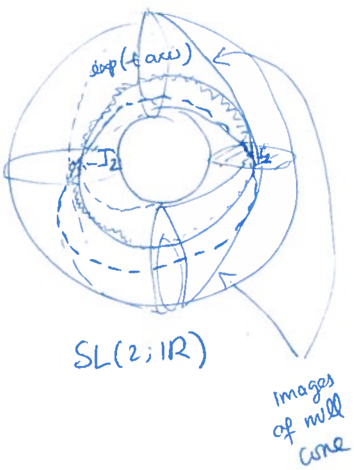
Topologically $SL(2;\mathbb{R}) \cong \mathbb{R}^2 \times S^1$

$\uparrow$ diffeomorphic to a solid torus

From exercises yesterday $\mathfrak{sl}(2;\mathbb{R}) = \{ X \in M_2(\mathbb{R}) : tr(X) = 0 \}$

$$\begin{matrix} \Psi \\ X \end{matrix} = \begin{pmatrix} r & s-t \\ s+t & -r \end{pmatrix}$$




minimal polynomial of $X$ is

$X^2 + \underbrace{(t^2-s^2-r^2)}I_2 = 0$

If this is $>0$ this will exponentiate like $e^{\lambda}$ $e^{-\lambda}$

$=0$ like a nilpotent matrix in ex. above

$<0$ like sine and cosine

exp doesn't hit everything.

negative definite matrices except for $-I$ are missed.

log is not well defined everywhere. It's also not 1-1.

If you understand this picture you understand a lot about what goes on with Lie groups in general.

Prop: Every differentiable homomorphism $\phi : (\mathbb{R},+) \to (G,\cdot)$ is of the form $\phi(t) = e^{tv}$ for $v \in \phi'(0) \in \mathfrak{g}$

$\uparrow$ ie. $\phi(s+t) = \phi(s)\phi(t)$

Proof: Use $\phi(s+t) = \phi(s)\phi(t)$ to see that $\phi(t)$ is an integrable curve of $X_v$ where $v = \phi'(0)$. (Exercise)

What about higher dimensions? For instance is $\exp: \mathfrak{g} \to G$ a homomorphism?

Well $\mathfrak{g}$ is abelian ~~then~~ Since it maps to a neighborhood of id in $G$ the connected component of $G$ would be abelian which is not true for $G = SL(2;\mathbb{R})$ for instance.

However we can measure how exp fails to be a homomorphism

$$\begin{aligned} \mathfrak{g} \times \mathfrak{g} &\longrightarrow G \\ (x,y) &\longrightarrow e^x e^y \end{aligned}$$

If $X, Y$ are small we can write $\log(e^X e^Y) = \overbrace{\nu_1(X,Y)}^{\text{1st order}} + \overbrace{\nu_2(X,Y)}^{\text{2nd order term}} + R_3(X,Y)$

We can plug in power series and collect terms to see what we get. The result is

$$\nu_1(X,Y) = X+Y$$
$$\nu_2(X,Y) = \tfrac{1}{2}(XY - YX) =: \tfrac{1}{2}[X,Y]$$

commutator bracket.

It is a little bit surprising that $[X,Y] \in \mathfrak{g}$, i.e. $\mathfrak{g}$ is closed under the commutator.

For instance symmetric trace 0 matrices are not closed under the commutator and indeed there is no Lie subgroup corresponding to this space. Eg symmetric matrices do not form a group.

Theorem: If $\mathfrak{g} \subset \mathfrak{gl}(n;\mathbb{R})$ is closed under the commutator bracket then there is an immersed submanifold of $GL(n;\mathbb{R})$ that is a subgroup of $GL(n;\mathbb{R})$ with tangent space $\mathfrak{g}$.

The reason for immersed is the following:

Example: $V = \begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & \sqrt{2} \\ & & -\sqrt{2} & 0 \end{pmatrix} \in \mathfrak{gl}(4;\mathbb{R})$

Copy of $\mathbb{R}$ in

$$\left\{ e^{tV} = \begin{pmatrix} \cos t & \sin t & & \\ -\sin t & \cos t & & \\ & & \cos\sqrt{2}t & \sin\sqrt{2}t \\ & & -\sin\sqrt{2}t & \cos\sqrt{2}t \end{pmatrix} \right\} \subset SO(4)$$

but it is not embedded. The closure is a 2 dimensional torus, the product of 2 circles

$$\begin{pmatrix} \cos\theta & \sin\theta & & \\ -\sin\theta & \cos\theta & & \\ & & \cos\alpha & \sin\alpha \\ & & -\sin\alpha & \cos\alpha \end{pmatrix}$$

(The line winds densely around a torus)

It now becomes an algebraic problem to find subspaces closed under $[\,,\,]$ and thus get a description of the Lie subgroups but that is a long story...

Prop: Given a differential homomorphism $\phi: H \to G$. If $\varphi = \phi'(I): \mathfrak{h} \to \mathfrak{g}$ then

$$\phi(e^{tv}) = e^{t\varphi(v)} \quad \text{for all } v \in \mathfrak{h} \quad (\text{these both have the right derivative})$$

and

$$\varphi([x,y]_{\mathfrak{h}}) = [\varphi(x), \varphi(y)]_{\mathfrak{g}}.$$

So one needs to understand maps which preserve the commutator.

Remark: The commutator satisfies the Jacobi identity $[[x,y],z] + [[y,z],x] + [[z,x],y] = 0$

(Just plug in the definition)

This motivates the following abstract definition

Definition: A Lie algebra is a vector space $\mathfrak{g}$ endowed with a bilinear bracket $[x,y] \in \mathfrak{g}$ for $x, y \in \mathfrak{g}$ satisfying

(i) $[x,y] = -[y,x]$

(ii) $[[x,y],z] + [[y,z],x] + [[z,x],y] = 0$

Conjectured by Cartan

Theorem (Ado) If $(\mathfrak{g}, \{,\})$ is a Lie algebra over $\mathbb{R}$ there's an $n$ and an injection

$$i: \mathfrak{g} \to \mathfrak{gl}(n; \mathbb{R})$$

such that $i(\{x,y\}) = [i(x), i(y)]$

In particular every Lie algebra $\mathfrak{g}$ is the Lie algebra of some matrix group. Unfortunately It is not true that every abstract Lie group (manifold with a multiplication) is a matrix group. even connected.

Exercise: There are only 2 distinct 2d Lie algebras over $\mathbb{R}$.

## Lecture 3

Exercise: Let $\mathfrak{g} = \mathbb{R}^3$, $\{x,y\} = x \times y$. Show $(\mathfrak{g}, \{,\})$ is a Lie algebra which is isomorphic to $\mathfrak{so}(3) = T_I SO(3)$

No 2d subspace of $\mathbb{R}^3$ is closed under $\times$ so this will show $SO(3)$ has no 2dim subgroups (which is not at all obvious)

These always are 1d subgroups. In $SO(3)$ these corresponds to one parameter groups of rotations and are all conjugates.

The Lie bracket of vector fields: on $\mathbb{R}^n$, say a vector field $V = \sum_{i=1}^n v^i(x) \frac{\partial}{\partial x_i}$ $\left(\text{convention} = v^i(x) \frac{\partial}{\partial x_i}\right)$

Can use it to differentiate a function

$$Vf = \sum_{i=1}^n v^i(x) \frac{\partial f}{\partial x_i} \quad f \in C^1(\mathbb{R}^n)$$

TÉCNICO LISBOA

There is a correspondence between vector fields and homogeneous $1^{st}$ order differential operators in the obvious way.

$V \cdot f$

This tells you how fast $f$ is changing when you flow in the direction of $V$

$W = w^i(x)\frac{\partial}{\partial x_i}$

$$V\cdot(W\cdot f) - W\cdot(V\cdot f) = \left(v^j \frac{\partial w^i}{\partial x_j} - w^j\frac{\partial v^i}{\partial x_j}\right)\frac{\partial f}{\partial x_j} =: [V,W]\cdot f$$

This is called the Lie bracket of the vector fields $V, W$.

Clearly $[V,W] = -[W,V]$

It's also true that $[\ ,\ ] : \mathrm{Vect}(\mathbb{R}^n) \times \mathrm{Vect}(\mathbb{R}^n) \longrightarrow \mathrm{Vect}(\mathbb{R}^n)$ satisfies the Jacobi

vector fields on $\mathbb{R}^n$

or $M$ any manifold

identity and hence makes $(\mathrm{Vect}(M), [\ ,\ ])$ into an $\infty$-dimensional Lie algebra.

Can think of $\mathrm{Vect}(M)$ either as $1^{st}$ order differential operators or as smoothly varying choices of tangent vectors over each point of the manifold.

Exercise: (i) $[X_V, X_W] = X_{[V,W]}$

commutator of $v$ and $w$

For $V, W \in M_n(\mathbb{R})$

(ii) If $Y_V(a) = Va$ we say $Y$ is right invariant. Check $[Y_V, Y_W] = -Y_{[V,W]}$

will be important later

The Lie bracket was introduced by Lie to study differential equations in n-variables.

Groups as manifestations of symmetries

Let $G$ be a Lie group with multiplication $\mu$, $M$ a manifold. A left action of $G$ on $M$ is a smooth map

$$\lambda : G \times M \longrightarrow M$$

that satisfies (i) $\lambda(e,m) = m$
(ii) $\lambda(a, \lambda(b,m)) = \lambda(ab, m)$

Abbreviate $\lambda(a,m) = a\cdot m$
Then (i) becomes $e\cdot m = m$
(ii) $a\cdot(b\cdot m) = (ab)\cdot m$

Examples: (i) $G = GL(n;\mathbb{R})$, $M = \mathbb{R}^n$. There is a natural action $A\cdot m := Am$

on $M = S_n(\mathbb{R})$ $\quad A\cdot s = A s\, {}^tA$

$n\times n$ symmetric matrices

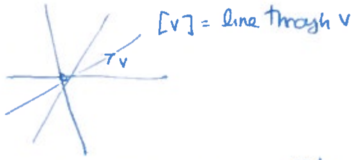
on $M = A_n(\mathbb{R})$ $\quad A\cdot J = A J\, {}^tA$

anti-symmetric matrices

(ii) $G = O(n)$  $M = S^{n-1} \subset \mathbb{R}^n$  $A \cdot v = Av$

(iii)
Let $\mathbb{R}P^{n-1}$ be the space of lines through $0$ in $\mathbb{R}^n$. This has a natural manifold structure.

in dim 2 →




Can take $G = SL(n;\mathbb{R})$, $M = \mathbb{R}P^{n-1}$ and $A \cdot [v] = [Av]$

$n=2$: $[v] = \begin{bmatrix} x \\ y \end{bmatrix}$  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$

If we cut out the horizontal, on the set of non horizontal lines can pick a coordinate by intersecting with the line $y=1$

On $\mathbb{R}P^1 \setminus [1:0]$ we have a coordinate $s\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \frac{x}{y}$

In terms of this coordinate $\begin{pmatrix} a & b \\ c & d \end{pmatrix} s = \frac{as+b}{cs+d}$  (Möbius transformation or fractional linear transformation)

This is not defined for matrices where $cs+d=0$ but it is defined for matrices near the identity

This was an important motivating example for Lie.

---

For any $v \in \mathfrak{g}$ let $e^{tv}$ be the 1-parameter subgroup and consider the induced flow on $M$ given by
$$\Phi_v(t) \cdot m = e^{tv} \cdot m = \lambda(e^{tv}, m)$$

There is a unique vector field on $M$, $\lambda_*(v) \in Vect(M)$ such that $\Phi_v$ is the time $t$ flow of $\lambda_*(v)$ so we get $\lambda_* : \mathfrak{g} \longrightarrow Vect(M)$

<u>Example</u>: $G = SL(2;\mathbb{R})$
$M = \mathbb{R} \subset \mathbb{R}P^1$ with coordinate $s$

$v = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$  $e^{tv} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$  $e^{tv} \cdot s = \frac{s}{1+ts}$ well defined for small enough $t$
$$\frac{s}{1+ts} = s - ts^2 + \cdots$$

So $\lambda_*(v) = -s^2 \frac{\partial}{\partial s}$

<u>Exercise</u>: $\lambda_* \begin{pmatrix} a_1 & a_0 \\ a_2 & -a_1 \end{pmatrix} = (a_0 + 2a_1 s - a_2 s^2)\frac{\partial}{\partial s}$ for any constants $a_0, a_1, a_2$.

<u>Theorem</u>: If $\lambda : G \times M \to M$ is a (local) left action of $G$ on $M$ then $\lambda_* : \mathfrak{g} \to Vect(M)$ is linear and $\lambda_*([v,w]) = -[\lambda_*(w), \lambda_*(v)]$

So it is a Lie algebra homomorphism except for the − sign.

In particular the image of $\lambda_*$ is a Lie algebra.

There's a lot of controversy about this − sign. There are ~~all~~ whole books written in the 20th century where the ~~sign~~ definition of Lie bracket was changed to fix this sign.

Corollary: span $\{ \frac{\partial}{\partial s}, s\frac{\partial}{\partial s}, s^2 \frac{\partial}{\partial s} \}$ is a Lie subalgebra of $Vect(\mathbb{R})$

Note: The f.d. Lie subalgebras of $Vect(\mathbb{R})$ were classified by Lie. This is the largest. This is an exercise in Exercise set 2 of the notes posted online.

The original idea was to do this for all $\mathbb{R}^n$ but this turns out to be too difficult. Up to now there are only partial results for $\mathbb{R}^4$.

ODE of Lie type:

You have probably learned how to solve $x'(t) = a_1(t)x(t)$ — linear homogeneous

$x'(t) = a(t)x(t) + b(t)$ ~~and maybe~~ — linear inhomogeneous

and maybe even $x'(t) = a_2(t)x(t)^2 + a_1(t)x(t) + a_0(t)$ — Ricatti equation [how to find a solution from a homogeneous one]

Each of this corresponds to a Lie subalgebra of $Vect(\mathbb{R})$!

$\{ x\frac{\partial}{\partial x} \}$

$\{ \frac{\partial}{\partial x}, x\frac{\partial}{\partial x} \}$

$\{ \frac{\partial}{\partial x}, x\frac{\partial}{\partial x}, x^2\frac{\partial}{\partial x} \}$

It was Lie who found this correspondence between subalgebras and certain types of differential equations.

Proposition: Let $\lambda : G \times M \to M$ ($G \leq GL(n;\mathbb{R})$) be a (local) left action and $A : \mathbb{R} \to \mathfrak{g}$ be a differentiable curve. Consider the matrix ODE $S'(t) = S(t)A(t)$ with $S(0) = Id$ [the fundamental solution associated to A]

Once we know the fundamental solution we obtain the others multiplying by a constant.

Consider a differential equation $\gamma'(t) = \lambda_*(A(t))(\gamma(t))$ [Lie equation associated to the action $\lambda$]

for a curve $\gamma : \mathbb{R} \to M$. Then $\gamma(t) = S(t) \cdot m$ is the unique solution with $\gamma(0) = m$.

Example: The Ricatti equation is the Lie equation associated to $SL(2;\mathbb{R})$ acting by linear fractional transformations.

The Ricatti equation is highly nonlinear while the Lie equation is linear for S so Lie's method transforms a non-linear equation into a linear equation provided you can get an embedding of the symmetries of the equation in matrices.

All the classical methods - variation of constants, reduction of order, ... turn out to be special cases of this.

This is an incredibly powerful method coupled with one other thing.

<u>Special case</u>: when $G$ can be taken to be upper triangular, i.e. $G \subseteq \left\{ \begin{bmatrix} * & \cdots & * \\ 0 & & * \end{bmatrix} \right\} \subseteq GL(n;\mathbb{R})$

In that case the equation becomes a system which can be solved iteratively by variation of parameters.

$$\begin{pmatrix} s_1'(t) \\ \vdots \\ s_n'(t) \end{pmatrix} = \begin{pmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ & \ddots & \vdots \\ 0 & & a_{nn}(t) \end{pmatrix} \begin{pmatrix} s_1(t) \\ \vdots \\ s_n(t) \end{pmatrix}$$

$$s_n'(t) = a_{nn}(t)\, s_n(t) \Rightarrow s(t) = s_0\, e^{\int a_{nn}(t)\,dt}$$

$$s_{n-1}'(t) = a_{n-1,n-1}(t)\, s_{n-1}(t) + \underbrace{a_{n-1,n}(t)\, s_n(t)}_{\text{known}}$$ Can solve with an integrating factor.

$$\vdots$$

This is called the <u>solvable case</u>

There is a test for when one can do this. Given a Lie algebra $\mathfrak{g}$ define

$\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}]$, $\mathfrak{g}_2 = [\mathfrak{g}_1, \mathfrak{g}_1]$, ...

This is a decreasing sequence of Lie algebras $\mathfrak{g} \supset \mathfrak{g}_1 \supset \mathfrak{g}_2 \supset \cdots$

<u>Theorem (Lie-Engel)</u> $\mathfrak{g}$ is isomorphic to a subalgebra of upper triangular complex matrices $\mathfrak{sl}(n;\mathbb{C})$ for some $n$ iff $\underbrace{\mathfrak{g}_k = 0 \text{ for some } k}_{\text{one says } \mathfrak{g} \text{ is solvable}}$.

In this case any Lie equation can be solved by quadratures.

$\mathfrak{sl}(2;\mathbb{R})$ and $\mathfrak{so}(3)$ are not solvable. All $\mathfrak{g}_k$ are equal to $\mathfrak{g}$. ~~[illegible]~~ So one can't solve the Riccati equation by quadratures. If we find one solution however one can restrict to the stabilizer of the symmetry group and then since ~~[illegible]~~ 2d Lie algebras are solvable.

The stabilizer is and we can then solve the equation.

For a long time people tried to solve $s' = a_0(t) + a_1(t)s + a_2(t)s^2 + a_3(t)s^3$ given one solution but in the end this is not possible because this ~~[illegible]~~ does not correspond to a Lie algebra.