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Theorem (Darboux) There is a map from 1-forms to 2-forms on any M called the exterior derivative such that in any coordinate system

$$d(a_i dx^i) = \frac{1}{2}\left(\frac{\partial a_i}{\partial x^k} - \frac{\partial a_k}{\partial x^i}\right) dx^k \wedge dx^i$$

It turns out that there is a formula relating this to the Lie bracket:

$$d\alpha(V,W) = V\cdot(\alpha(W)) - W\cdot(\alpha(V)) - \alpha([V,W])$$

This does not depend on the choice of coordinates but it does depend on knowing what $[\,,\,]$ is.

This generalizes to 3, 4-forms and so on. The generalization to arbitrary dimensions is due to Goursat and leads to a Stokes theorem in all dimensions.

It's natural to think of Lagrangians as functions on the tangent bundle. The EL equations should be written geometrically on TM. ~~At least~~ Next time we'll see that via a magical transformation due to Legendre these can be transported to T^*M leading to symplectic geometry.

Lecture 5

Linear algebra: V a vector space over $\mathbb{R}$

$$\beta : \underbrace{V \times \cdots \times V}_{k \text{ times}} \longrightarrow \mathbb{R} \qquad k\text{-multilinear}$$

You are familiar with the case $k=2$ e.g.
(i) $\beta(x,y) = \beta(y,x)$ symmetric
(ii) $\beta(x,y) = -\beta(y,x)$ anti-symmetric (or skew-symmetric)

Ex: $V = \mathbb{R}^n$ $\quad \beta(x,y) = {}^t x B y$, $B \in M_n(\mathbb{R})$ in case (i) B is symmetric and in case (ii) it is anti-symmetric

For $k=2$, β is non-degenerate if $\beta(x,y)=0 \ \forall y \Rightarrow x=0 \iff \det B \neq 0$

If $B = -{}^tB$ and B is non-degenerate then n is even (because $\det B = (-1)^n \det B$). When n is even it is easy to write examples, e.g. $B = \begin{pmatrix} 0 & I_{n/2} \\ -I_{n/2} & 0 \end{pmatrix}$. Interestingly over any field of char $\neq 2$ for fixed n even there is only one skew-symmetric non-degenerate form up to isomorphism (this is different from the symmetric case where even over $\mathbb{R}$ there is an invariant — the signature = # of negative eigenvalues of the matrix).

(V,β) with β skew, non-degenerate is called a symplectic vector space.

One more notion: Contraction: If β is k-multilinear then for $x \in V$ we define

$$(x \lrcorner \beta)(y_2, \dots, y_k) = \beta(x, y_2, \dots, y_k)$$

$\uparrow$ $(k-1)$-multilinear.

Remark: For $k=2$ and β is non-degenerate then $x \lrcorner \beta = 0$ iff $x=0$ because

$$\begin{aligned} V &\longrightarrow V^* \\ x &\longmapsto x \lrcorner \beta \end{aligned} \quad \text{is an isomorphism}$$

On manifolds: A k-form is $\beta: Vect(M) \times \dots \times Vect(M) \to C^\infty(M)$ multi-linear over $C^\infty(M)$ and skew-symmetric

$$p \in M \qquad \beta_p : T_pM \times \dots \times T_pM \longrightarrow \mathbb{R}$$

$\Rightarrow$ If $f: M \to N$ and β is a k-form on N then we can define

$$(f^*\beta)(v_1, \dots, v_k) = \beta(f'(v_1), \dots, f'(v_k))$$

where $f': TM \to TN$ is the derivative. If X is a vector field on M and β is a k-form on M then we can let $\Phi_t : M \to M$ be the time t flow of X_t. and take

$$\mathcal{L}_X \beta \overset{def}{=} \lim_{t \to 0} \frac{\Phi_t^* \beta - \beta}{t}$$

not hard to show this is a well defined k-form

Lie derivative of β with respect to X.

Easy to show: $\mathcal{L}_X \beta = 0$ iff $\Phi_t^* \beta = 0$ for all t, i.e. β is fixed by the flow of X.

Cartan formula: $\boxed{\mathcal{L}_X \beta = X \lrcorner d\beta + d(X \lrcorner \beta)}$

The ~~????~~ anti-commutator of contraction and exterior derivative gives the Lie derivative.

Here $(X \lrcorner \beta)_p = \beta_p(X_p, \cdot, \cdot, \cdot)$

These formulas are generalizations of the basic formulas from Calculus and will need to use them for our geometric formulation.

Facts about tangent bundle TM: It is natural, i.e. if $f: M \to N$ is a smooth map then

$f': TM \to TN$ is also a smooth map.

If X is a vector field on M with local flow $\Phi_t : M \to M$ then $\Phi_t' : TM \to TM$ is the local flow of a vector field X' on TM

Exercise: $v \in TV \subset TM$, $x \in V \subset M$. If $X = a^i(x) \frac{\partial}{\partial x_i}$ then $X' = a^i(x) \frac{\partial}{\partial x^i} + v^i \frac{\partial a^j}{\partial x^i} \frac{\partial}{\partial v^j}$

Let $L: TM \to \mathbb{R}$ be our Lagrangian. $\gamma: [a,b] \to M$ C^1 curve and $A(\gamma) = \int_a^b L(\dot\gamma(t))\, dt$

Basic Proposition: Given $L: TM \to \mathbb{R}$ there exists a unique function $E_L : TM \to \mathbb{R}$ and a 1-form ω_L on TM so that in any local coordinate system (x, v)

$$E_L = v^i \frac{\partial L}{\partial v^i} - L \qquad \omega_L = \frac{\partial L}{\partial v^i} dx^i$$

So these (expressions defined in local coord) are independent of the choice of coordinates.

These are interesting because a C^2-curve $\gamma : [a,b] \to M$ is L-critical iff the tangent curve $\dot\gamma : [a,b] \to TM$ satisfies

$$(*) \qquad \ddot\gamma(t) \lrcorner \, d\omega_L = -(dE_L)_{\dot\gamma(t)}$$

this confuses people who have taken Diff. Geom.
They'll say you can't define acceleration without a connection. But this is the derivative of a curve on TM.

$$\ddot\gamma(t) \in T_{\dot\gamma(t)}(TM)$$

Exercise: Write down $(*)$ in local coordinates and check these are the EL equations for γ to be L-critical.

The beauty is that these can be expressed in terms of this 1-form and function on TM.

TM is a union of vector spaces. Each of them has a canonical radial vector field. E_L is just the derivative of L wrt this canonical vector field $- L$ (so the fact that E_L is well defined is trivial). ~~Also~~ The fact that ~~this~~ ω_L is well defined is less trivial.

If $y = F(x)$ were another coordinate system on U then $w = F'(x) v$
As we saw last time

$$dy = F'(x)\, dx$$
$$dw = G(x,v)\, dx + F'(x)\, dv$$

$$dL = L_x\, dx + L_v\, dv = L_y\, dy + L_w\, dw$$
$$= L_y (F'(x) dx) + L_w (G(x,v) dx + F'(x) dv)$$

So $L_v = L_w F'(x)$ (where $\omega_L = L_v dx$)

$$\Downarrow$$
$$\omega_L = L_v dx = L_w F'(x) dx = L_w dy \quad \text{and } \omega_L \text{ is well defined.}$$

Corollary: The function E_L is constant on $\dot\gamma : [a,b] \to TM$ whenever $\gamma : [a,b] \to TM$ is L-critical

Proof: $\frac{d}{dt}(E_L(\dot\gamma)) = (dE_L)_{\dot\gamma}(\ddot\gamma) = -(\ddot\gamma \lrcorner \, d\omega_L)(\ddot\gamma) = -d\omega_L(\ddot\gamma, \ddot\gamma) = 0$ $\square$

because $d\omega_L$ is skew

Example: $\ddot x = -\text{grad}\, V(x)$ are the EL equations for ~~the~~ $A(\gamma) = \int \frac{1}{2} |\dot\gamma|^2 - V(\gamma)\, dt$

need a size for the velocity so need a Riemannian metric to make sense of this on a manifold

(ie $L = \frac{1}{2}|v|^2 - V(x)$). Hence $E_L = \frac{1}{2}|v|^2 + V(x)$

<u>What about symmetries?</u>

<u>Definition</u>: A vector field X on M is an <u>infinitesimal symmetry</u> of $L : TM \to \mathbb{R}$ if the flow of X' on TM preserves L (i.e. leaves L invariant)

The idea is that such a vector field generates a ~~1-dimensional~~ 1 parameter symmetry group and the vector field is obtained from the action by differentiating.

Note that if X is an infinitesimal symmetry of L then the flow of X' leaves E_L and ω_L fixed, i.e. $\mathcal{L}_{X'} E_L = 0$ and $\mathcal{L}_{X'} \omega_L = 0$. Therefore the flow will carry solutions of the EL equations to other solutions.

<u>Theorem</u>: If X is an infinitesimal symmetry of L then the function $\omega_L(X')$ is constant on all $\ddot{\gamma} : [a,b] \to TM$ ~~such that~~ such that γ is L-critical.

One says $\omega_L(X')$ is a first integral of the EL-equations

<u>Proof</u>: $d(\omega_L(X'))(\ddot{\gamma}) = (d(X' \lrcorner\, \omega_L))(\ddot{\gamma}) \underset{\text{by Cartan formula}}{=} (\underbrace{\mathcal{L}_{X'}\omega_L}_{\substack{= 0 \\ \text{by hypothesis}}} - X' \lrcorner\, d\omega_L)(\ddot{\gamma}) =$

$$= -(X' \lrcorner\, d\omega_L)(\ddot{\gamma}) = -d\omega_L(X', \ddot{\gamma}) = d\omega_L(\ddot{\gamma}, X') = (\ddot{\gamma} \lrcorner\, d\omega_L)(X') = (-dE_L)(X') = -\mathcal{L}_{X'} E_L \underset{\text{by hypothesis}}{=} 0$$

$\square$

Nowadays this is called Noether's Theorem but in formulas it was known before Noether. Noether first observed that this was a general procedure for turning symmetries into conserved quantities and vice-versa.

Noether's Theorem is a vast generalization that applies to variational problems in all dimensions (Noether is perhaps best known for her later work in algebra but at the beginning of the 20th century she worked in Math. Physics)

The equation $\ddot{\gamma} \lrcorner\, d\omega_L = E_L$ can be abstracted considerably. To generalize this we need a manifold X, a 2-form Ω on X ~~non-degenerate~~ and a function $H : X \to \mathbb{R}$.

If Ω is non-degenerate we can write a vector field X_H by $X_H \lrcorner\, \Omega = -dH$

The letter H honors Hamilton and H is usually called the Hamiltonian. The arguments we saw again show that symmetries of H will produce conserved constants. All that is required is that $d\omega = 0$ (this locally says ω is d of a 1-form but not nec. globally). This is called a ~~symplec~~

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symplectic structure on the manifold X.

All that is needed to tie conserved quantities to symmetries of the function is this closed non-degenerate 2-form.

This would not be very useful if those structures were rare but in fact these structures are everywhere in Mathematics.

Symplectic geometry is nowadays much better understood via the application of elliptic PDE methods in the work of Gromov, Hofer, McDuff, Salamon.

Note: One can write Lagrangians that have fewer symmetries than the equations. There are instances of this "hidden symmetry" - Toda lattice, non-linear σ-model.

In those cases (Guillemin-Sternberg) the system you are looking at are shadows of more complicated systems which have more symmetry. This is not yet completely understood.

One might think this is just some reformulation of the fact that if we have a symmetry one can just eliminate one variable.

In the variational case when we fix a conserved quantity we reduce to a level set. We still have the symmetry so we can reduce the number of variables by twice the amount. This is called the Marsden-Weinstein reduction method and is the reason why variational problems are so much easier.