TÉCNICO LISBOA

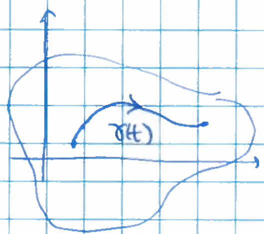
Robert Bryant - lecture 4

We have been talking about differential equations but not so much where they come from. Today I want to talk about a source, namely variational problems and deduce the Euler-Lagrange equations. A choice of good coordinates leads to symplectic geometry.

Variational problems:

Consider paths in $\mathbb{R}^n$ or some domain

$$\gamma : [a,b] \longrightarrow \mathbb{R}^n$$

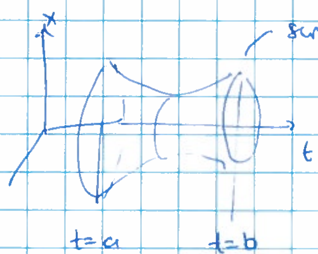


We are interested on functions that depend on position and velocity

$$L : \underset{\substack{x \\ \| \\ x^i \\ \text{position}}}{\mathbb{R}^n} \times \underset{\substack{v \\ \| \\ v^i \\ \text{velocity}}}{\mathbb{R}^n} \longrightarrow \mathbb{R} \qquad A(\gamma) = \int_a^b L(\gamma(t), \dot\gamma(t))\, dt$$

L is usually called the Lagrangian (nothing to do with Lagrangian subspaces in symplectic geometry)

Examples: (a)

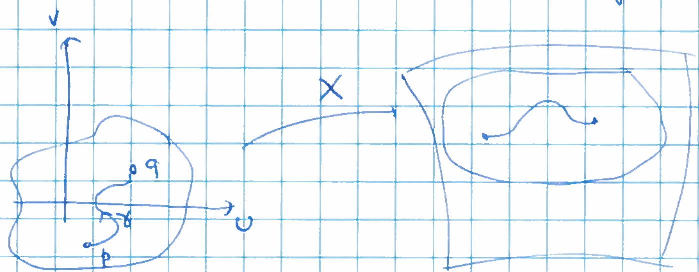


surface of revolution generated by $(t, y(t))$

$$\text{Area}(y) = \int_a^b 2\pi y(t) \sqrt{1+y'(t)^2}\, dt$$

If we are interested in producing such a surface of least area then we see it is a special case of the previous problem

(b)



shortest path joining 2 points on the surface parametrized by X

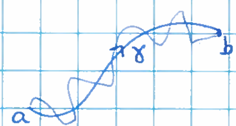
$$A(\gamma) = \int_0^1 |dX(\dot\gamma)|\, dt \qquad \text{is the length of the image curve}$$

This is called the geodesic problem which also makes sense in n dimensions.

$$dX = X_u\, du + X_v\, dv \quad \text{so} \quad |dX(\dot\gamma)|^2 = (X_u \cdot X_u)(\gamma)\, \dot u^2 + 2(X_u \cdot X_v)(\gamma)\, \dot u \dot v + (X_v \cdot X_v)(\gamma)(\dot v)^2$$

so this is again an instance of the problem above.

We say that γ is an <u>L-critical path</u> if for <u>any</u> variation



$$\Gamma : [a,b] \times (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}^n$$

$$\Gamma(t,s) = \begin{cases} \gamma(t) & \text{if } s = 0 \\ \gamma(a) & \text{if } t = a \\ \gamma(b) & \text{if } t = b \end{cases}$$

(ie. they all start at the same points and end)

We have that
$$0 = \frac{d}{ds}\Big|_{s=0} \left(A(\Gamma(\cdot, s)) \right)$$

curve obtained by holding s fixed

This will certainly be true if γ is a local minimum or a local maximum

Let $h : [a,b] \longrightarrow \mathbb{R}^n$ with $h(a) = h(b) = 0$ and
$$\Gamma(t,s) = \gamma(t) + s h(t)$$

We'll compute the above derivative in this specific example

$$\frac{d}{ds}\Big|_{s=0} \int_a^b L\left(\gamma(t) + sh(t), \dot\gamma(t) + s\dot h(t)\right) dt$$

(assuming everything is smooth enough ~~...~~ so that we can differentiate under the integral sign and applying the chain rule)

$$= \int_a^b \left(\frac{\partial L}{\partial x^k}(\gamma, \dot\gamma)\, h^k(t) + \frac{\partial L}{\partial v^k}(\gamma, \dot\gamma)\, \dot h^k(t) \right) dt$$

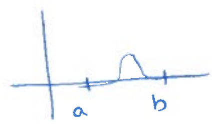
$$= \int_a^b \frac{\partial L}{\partial x^k}(\gamma, \dot\gamma)\, h^k(t) - \frac{d}{dt}\left(\frac{\partial L}{\partial v^k}(\gamma, \dot\gamma) \right) h^k \, dt$$

(integrating by parts. There is no boundary term because h vanishes at the end points)

$$= \int_a^b \left(\frac{\partial L}{\partial x^k}(\gamma, \dot\gamma) - \frac{d}{dt}\left(\frac{\partial L}{\partial v^k}(\gamma, \dot\gamma) \right) \right) h^k \, dt$$

This is supposed to be 0 no matter what h (vanishing at the end points) is.

Let $\varphi : [a,b] \to [0, \infty[$ be any function with $\varphi(a) = \varphi(b) = 0$ and set
$$h^k = \varphi \left(\frac{\partial L}{\partial x^k}(\gamma, \dot\gamma) - \frac{d}{dt}\left(\frac{\partial L}{\partial v^k}(\gamma, \dot\gamma) \right) \right)$$



when we plug this in we get $\displaystyle\int_a^b \varphi \sum_k \left(\frac{\partial L}{\partial x^k}(\gamma, \dot\gamma) - \frac{d}{dt}\left(\frac{\partial L}{\partial v^k}(\gamma, \dot\gamma) \right) \right)^2$

If this does not vanish at some point we could find φ so that this would be positive so we must have

<u>Euler-Lagrange</u>. A necessary condition to be L-critical is
$$\frac{\partial L}{\partial x^k}(\gamma, \dot\gamma) - \frac{d}{dt}\left(\frac{\partial L}{\partial v^k}(\gamma, \dot\gamma) \right) = 0$$

Although these were discovered by Euler they were not easy to find in his writings. Lagrange was the first to show their importance but he attributed them to Euler.

One can also show that a solution of these equations is necessarily L-critical.

The Euler-Lagrange equations are second order equations for the unknown functions y:

$$\frac{\partial L}{\partial x^k}(y,\dot y) - \frac{\partial^2 L}{\partial v^k \partial x^j}(y,\dot y)\,\dot y^j - \frac{\partial^2 L}{\partial v^k \partial v^j}(y,\dot y)\,\ddot y^j = 0$$

(using Einstein summation convention - sum over repeated indices)

Examples: (1) $L = g_{ij}(x) v^i v^j$ $\quad L$ is a Riemannian metric

Plugging this into the above equations we get

$$\frac{\partial g_{ij}}{\partial x^k}(y)\,\dot y^i \dot y^j - \frac{d}{dt}\left(2 g_{ik}(y)\,\dot y^i\right) = 0$$

$$\frac{\partial g_{ij}}{\partial x^k}(y)\,\dot y^i \dot y^j = 2\frac{\partial g_{ik}}{\partial x^j}\dot y^i \dot y^j + 2 g_{ik}(y)\,\ddot y^i = 0$$

If g is positive definite (g_{ik}) is invertible and we can solve for the $\ddot y^k$.

$$\ddot y^i = -\Gamma^i_{kj}\,\dot y^k \dot y^j$$

Exercise: $\Gamma^i_{jk} = \frac{1}{2} g^{il}\left(\frac{\partial g_{kl}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\right)$ $\quad$ where $(g^{ij}) = (g_{ij})^{-1}$

These expressions show up everywhere in Riemannian geometry and it is interesting they arise in this simple way from the Euler-Lagrange equations.

Exercise: If $L = \frac{1}{2}|v|^2 - V(x) = \frac{1}{2} v\cdot v - V(x)$ where $V: \mathbb{R}^n \to \mathbb{R}$ then the EL equations become $\ddot y = -\operatorname{grad} V(y)$ (equations of motion in a conservative force field)

So this equation of motion is a variational equation (the name for equations which can be put in this form). There are lots of examples - practically anything in physics. People say that a good physical theory is the same as a good choice of Lagrangian.

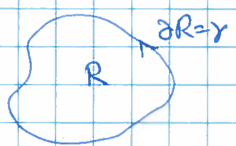
You can imagine that there are versions where the functions have more than one independent variable. That is what is involved in M-theory and so on. When people say they don't know what a theory is they typically mean they don't know what the Lagrangian is.

(2) Suppose $L = a_i(x) v^i$ $\quad a_i: \mathbb{R}^n \to \mathbb{R}$. Then you find that the EL equations are simply

$$\left(\frac{\partial a_i}{\partial x^k}(y) - \frac{\partial a_k}{\partial x^i}(y)\right)\dot y^i = 0$$

this is a linear equation
it should ring a bell from vector calculus.

Recall Green's Theorem:

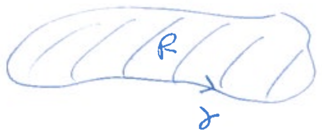


$$\int_\gamma P\,dx + Q\,dy = \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

When $n=2$ the equations above say that if we want the variation of γ to be a critical point then it should integrate to 0 over any curve that bounds a surface.

Green's Theorem works for any dimension.

Given a surface R in space with boundary γ

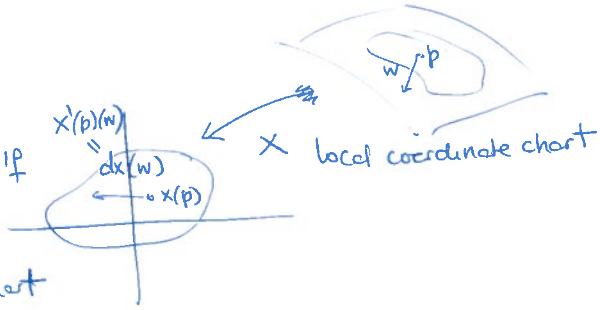


$$\int_\gamma a^i dx^i = \frac{1}{2}\int_R \underbrace{\left(\frac{\partial a_k}{\partial x_i} - \frac{\partial a_i}{\partial x_k}\right) dx_i \wedge dx_k}_{\text{2-forms}}$$

sum over i, k

Need to tell you how to intepret the above expressions and I will do this in the context of an arbitrary manifold.

Geometry of $TM = \bigcup_{x\in M} T_xM$



Remark: TM is a manifold. If

$x: U \to \mathbb{R}^n$ is a local coord chart

then $(x, dx): TU \to \mathbb{R}^n \times \mathbb{R}^n$

is a coordinate chart for TU

If $y = F(x)$ for y a different coordinate chart then $dy = F'(x)\,dx$ (chain rule)

The rule for relating (x, v) with (y, w) is $\begin{cases} dy = F'(x)dx \\ dw = F'(x)dv + G(x,v)dx \end{cases}$ for some $G(x,v)$

think of as "dx" secretly

change of coordinates in TM

A 1-form is a function α on vector fields which is linear over smooth functions

$$\alpha(fV + gW) = f\alpha(V) + g\alpha(W)$$

for all $f, g \in C^\infty(M)$ + $V, W \in \mathrm{Vect}(M)$. One can think of these as functions on TM which are linear on each tangent space T_xM.

A 2-form is a function on pairs of vector fields on M such that

(i) $\beta(V,W) = -\beta(W,V)$ (skew-symmetry)

(ii) β is $C^\infty(M)$-linear in each slot separately.

Example: On $M = \mathbb{R}^n$ with coordinates x^i

$$dx^i \wedge dx^j \left(v^k \frac{\partial}{\partial x^k}, v^\ell \frac{\partial}{\partial x^\ell}\right) = v^i w^j - v^j w^i$$

is a 2-form on $\mathbb{R}^n$

(signed area of the projection of the parallelogram spanned by v, w onto the i,j plane)

There are $\binom{n}{2}$ of these 2-forms.

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Theorem (Darboux) There is a map from 1-forms to 2-forms on any M called the exterior derivative such that in any coordinate system

$$d(a_i dx^i) = \frac{1}{2}\left(\frac{\partial a_i}{\partial x^k} - \frac{\partial a_k}{\partial x^i}\right) dx^k \wedge dx^i$$

It turns out that there is a formula relating this to the Lie bracket:

$$d\alpha(V,W) = V\cdot(\alpha(W)) - W\cdot(\alpha(V)) - \alpha([V,W])$$

This does not depend on the choice of coordinates but it does depend on knowing what $[\,,\,]$ is.

This generalizes to 3,4-forms and so on. The generalization to arbitrary dimensions is due to Goursat and leads to a Stokes theorem in all dimensions.

It's natural to think of Lagrangians as functions on the tangent bundle. The EL equations should be written geometrically on TM. ~~At least~~ Next time we'll see that via a magical transformation due to Legendre these can be transported to T^*M leading to symplectic geometry.