Robert Bryant - An Introduction to Lie groups, symmetries and symplectic geometry.

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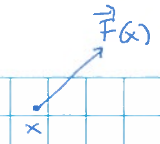
Motivation

Problem: Solving motion in a force field



$F: \mathbb{R}^n \to \mathbb{R}^n$ say smooth

Want the motion of an object of unit mass acted on by this force: $\ddot{x} = F(x(t))$.
What can we say about the trajectories

If F is completely arbitrary there's very little one can say about this. In special circumstances one can say more.
If $F = \nabla V$ for some $V: \mathbb{R}^n \to \mathbb{R}$ then $\ddot{x}(t) = -\nabla V(x)$

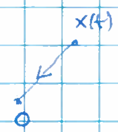
and $\frac{d}{dt}(|\dot{x}|^2) = 2\dot{x}\ddot{x} = -2\nabla V(x)\cdot\dot{x} = \frac{d}{dt}(2V(x))$ (F is called a conservative force field)

So $E = \frac{1}{2}|\dot{x}|^2 + V(x)$ is constant along the trajectory

(kinetic energy, potential energy)

We can specialise even more. If $V(x) = \frac{1}{2}v(|x|^2)$ is invariant under rotation

Then $\ddot{x} = -v'(|x|^2)\,x$ (central force field)



We now have another conservation law (assume n=3)

$$\frac{d}{dt}(x\times\dot{x}) = \dot{x}\times\dot{x} + x\times\ddot{x} = \dot{x}\times\dot{x} + x\times(-v'(|x|^2)\,x) = 0+0=0$$

So $\boxed{x\times\dot{x} = \mu}$ is constant. This is called conservation of angular momentum (which is what μ is called)

$\boxed{|\dot{x}|^2 + v(|x|^2) = 2E}$ ← tell us a lot about motion in a central force field.

$(x,\dot{x})$ is a curve in $\mathbb{R}^3\times\mathbb{R}^3 = \mathbb{R}^6$ subject to the above 4 relations. So the curve is constrained to live on a surface.
So the differential equations in 6 variables is secretly one in 2 variables which is a considerable savings.

Eventually it was realized that there was a general principle underlying this coming from the symmetry of the equations. That is what I will try to describe

Symmetry of the equations: what changes can we make to the equations that preserve it.
Origin is clearly a special point.
$\ddot{x}$ colinear with x means we should examine some linear properties in $\mathbb{R}^3$. Namely the set of rotations in $\mathbb{R}^3$:

A map $R: \mathbb{R}^3 \to \mathbb{R}^3$ is a rotation if
(i) $Rx\cdot Ry = x\cdot y$ for all $x,y\in\mathbb{R}^3$ $\Leftrightarrow$ ${}^tR\,R = Id$
(ii) R preserves orientation of $\mathbb{R}^3$ $\Leftrightarrow$ $\det R = 1$ (gets rid of the reflections)

This implies that $(Rx)\times(Ry) = R(x\times y)$, $|Rx|^2 = R|x|^2$

If x is a solution of $\ddot{x} = -v'(|x|^2)\,x$ then so is Rx (R comes out of both sides linearly)
Moreover $R(x\times\dot{x}) = Rx\times R\dot{x}$ so R acts on the angular momentum in a nice way. This gives a hint that there is some relation between the rotations and angular momentum.

Note: Reflections would flip angular momentum but we'll see later they would not yield any additional constants of motion.

There's nothing special about 2 and 3 in the definitions, that André gave. We'll see that rotations form a 3 dimensional object in 9 dimensional space. It is that 3 which explains the 3 conserved quantities of the angular momentum.

The energy conservation comes from the invariance of the equations under translations in time.

In order to make all of this precise we'll need to examine the geometry of space of rotations and to use calculus of variations to establish the link with conserved quantities.

|| Geometry of rotations, calculus.
Notion of (matrix) group

Differentiable map (smoothly) $f: U \subseteq \mathbb{R}^m \to \mathbb{R}^n$

⇓

Taylor expansion $f(a+h) = f(a) + f'(a)h + R_2(a,h)$ ← vanishes quadratically with h; $f'(a): \mathbb{R}^m \to \mathbb{R}^n$ linear, $\lim_{|h|\to 0} \frac{R_2(a,h)}{|h|} = 0$

a is a critical point of f if $f'(a): \mathbb{R}^m \to \mathbb{R}^n$ is not surjective.

$b \in \mathbb{R}^m$ is a critical value if there is no critical point in $f^{-1}(b)$.

Implicit function theorem: if $b \in \mathbb{R}^n$ is not a critical value of f then $f^{-1}(b) \subset U$ is a submanifold of dimension m−n. In fact $T_a f^{-1}(b) = \operatorname{Ker} f'(a)$ for all $a \in f^{-1}(b)$.

As long as μ and E are not a critical value of the functions above the level sets are a surface in $\mathbb{R}^6$.

Exercise: $f(x,y) = x^2 - y^2 + (x^2+y^2)^2$

Find the critical values, points. There are 3 critical pts + 2 critical values. The remaining level sets are smooth curves. The pre images of the critical values are not smooth.

Example: let $f(A) = A^T A$ and regard f as a map $f: M_n(\mathbb{R}) \to S_n(\mathbb{R})$ ← symmetric n×n matrices

$$f(A+H) = (A+H)^T(A+H) = f(A) + \underbrace{{}^tAH + {}^tHA}_{f'(A)\cdot H} + \underbrace{{}^tHH}_{R_2(A,H)}$$

$f'(A): M_n(\mathbb{R}) \to S_n(\mathbb{R})$

I claim the identity is a regular value of f, i.e. if ${}^tAA = Id$ (i.e. $A \in O(n) = \{A \in M_n(\mathbb{R}) : {}^tAA = Id\}$)

Then $f'(A): M_n(\mathbb{R}) \to S_n(\mathbb{R})$ is surjective

Given $B = B^t \in S_n(\mathbb{R})$ just let $H = \frac{1}{2}AB$. Then $f'(A)H = {}^tA\frac{1}{2}AB + {}^t(\frac{1}{2}AB)A = B$

So O(n) the orthogonal matrices are a submanifold of dim ~~[illegible]~~ $n^2 - \frac{1}{2}n(n+1) = \frac{1}{2}n(n-1)$ [for n=3, O(has dimension

Note that O(n) is closed. It is also compact because tr ${}^tAA = n$.

Exercise: (1) Show that $SL(n;\mathbb{R}) = \{A \in M_n(\mathbb{R}) : \det(A) = 1\}$ is a submanifold of $M_n(\mathbb{R})$ and compute its dimension

(2) let $J_n = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \in M_{2n}(\mathbb{R})$ and let $Sp(n;\mathbb{R}) = \{A \in M_{2n}(\mathbb{R}) : {}^tAJ_nA = J_n\}$. Show this is a submanifold of $M_{2n}(\mathbb{R})$.

↑ Some books say $Sp(2n;\mathbb{R})$

These objects ($O(n)$, $SL(n;\mathbb{R})$, $Sp(n;\mathbb{R})$) have more interesting algebraic properties than just being submanifolds - they are groups. i.e. they are nonempty, closed under multiplication and inverse

↑ check this.

Hint: $L_a: G \to G$, $g \mapsto ag$

i.e. they are matrix groups

Exercise: Compute $T_{I_n}G$ for $G = O(n), SL(n;\mathbb{R}), Sp(n;\mathbb{R})$ and show that $T_aG = \{av \mid v \in T_{I_n}G\}$