Problem 1. We compute a discrete ideal image $u[i,j]$ using an iterative method (Jacobi or Gauss-Seidel), where $u[i,j]$ is the intensity of gray for the pixel at the point $p_{ij} = (ih, jh)$ ($h > 0$ is the pixel size). Being an iterative method, we have computed $u[i,j,k]$, where k represents the k-th iteration.

Find a new step $\overline{h}$ and a new continuous variable $t = \overline{h}k$ such that the function of three variables $u(x_1, x_2, t)$ satisfies a PDE (as h and $\overline{h}$ tend to 0). Which is this PDE when using Jacobi's method, and which is for Gauss-Seidel's method? [If necessary, check in Internet what are the Jacobi and Gauss-Seidel methods.]

Problem 2. Let $H \subset \mathbb{R}^n$ be a bounded regular open set such that $\overline{B}_\varepsilon(0) = \{x \in \mathbb{R}^n : |x| \le \varepsilon\} \subset H$. Let us consider the open set $\Omega := H \setminus \overline{B}_\varepsilon(0)$ and a solution $u \in C^2(\overline{\Omega})$ of

$$\begin{cases} \Delta u + a(x) u_{x_1} = 1 & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \nu} = -1 & \text{on} \quad \partial H, \\ u = x_2 & \text{on} \quad \partial B_\varepsilon(0), \end{cases}$$

where a continuous function in $\overline{\Omega}$, ν is the exterior normal vector to Ω, and $\frac{\partial u}{\partial \nu} := \nabla u \cdot \nu$ is the directional derivative of u in the direction of ν (we call it the normal derivative of u). Find the exact value of $\max_{\overline{\Omega}} u$ and prove your answer.

Problem 3. The intensity of gray $u = u(x,y)$ of a black and white continuous image at $(x,y) \in \overline{Q} = [0,1]^2$ satisfies the problem

$$\begin{cases} -\Delta u + 4\lambda u = f, & (x,y) \in Q = (0,1)^2, \\ u = g, & (x,y) \in \partial Q, \end{cases}$$

where $\lambda \in \mathbb{R}$ is a constant and f and g are given functions (in a neighborhood of $\overline{Q}$ if necessary).

(i) Find the discrete version (with $m \times m$ pixels) for this problem.

(ii) Prove that the discrete problem admits a unique solution if $|\lambda|$ is small enough.

(iii) Prove that the discrete problem admits a unique solution if $\lambda > 0$.

(iv) What can you say about the existence of the solution when $\lambda < 0$?