Mathematical Modeling in Hydrodynamics; A. Nachbin

Problem Set 1: warm-up with ODEs

(1): Take the scalar ordinary differential equation (ODE)

$$\frac{dY}{dt}(t) = f(Y(t),t)), \quad Y(0) = Y_0.$$

(A) Deduce the methods presented in class using Taylor series expansions.

(B) Deduce the methods presented in class using numerical integration.

(C) Let

$$\frac{dY}{dt}(t) = \lambda Y(t), \quad Y(0) = Y_0, \quad Re(\lambda) < 0.$$

Discuss the stability of the three numerical methods.

(D) Discuss the properties of each method for a general complex λ.

(E) Discuss the stability of

$$\frac{d\vec{Y}}{dt}(t) = A\vec{Y}(t)), \vec{Y}(0) = \vec{Y}_0.$$

where A is a 2x2 constant matrix.

(2): The Lotka-Volterra model for population dynamics (where k_j are constants):

$$\frac{dN_1}{dt}(t) = k_1 N_1 - k_2 N_1 N_2,$$
$$\frac{dN_2}{dt}(t) = k_3 N_1 N_2 - k_4 N_2,$$

where the initial population densities are $N_1(0) = N_1^0$, $N_2(0) = N_2^0$ (individuals/area).

(A) Linearize the system.

(B) What are the critical/stationary points N_1^*, N_2^*?

(C) Let $N_j = N_j^* + \varepsilon \tilde{N}_j(t)$, $j = 1, 2$, $\varepsilon \ll 1$. Linearize the system.

(D) Describe in words the behavior near each critical point. Which numerical method from problem (1) would you use to solve the problem near the non-trivial critical point?