Problem 1. Let $b \in \mathbb{R}$ be a constant. Consider the equation

$$u_{xx} + bu_x - u = 0, \qquad x \in \mathbb{R}.$$

Find a probabilistic interpretation of the problem. That is, given an increment $h > 0$, find the probabilities of moving towards right and towards left, deducing the PDE from them. (Hint: at any time, there might be a probability of the particle being absorbed by the media, and thus dissapearing).

Problem 2. Let Ω be a rectangle in the plane and suppose that it is discretized (or filled) by $k \times m$ squares, all of them with sides of a given length $h > 0$. Thus, the rectangle is covered by k columns and m rows of squares of size $h \times h$. Consider the discrete version of our random walk game. That is, we have a function $v = v(i, j)$ assigning the exiting probability for the square in position (i, j) (all i, j, k, m are nonnegative integers). In particular:

(a) For every square touching the four boundaries of the rectangle, we know the value of v, being equal to 0 or 1 depending if the square corresponds to a closed or open part of the boundary.

(b) For the rest of the squares, the value of v at a square should be the average of the values of v at the squares in the east, west, north, and south (by conditioned probabilities).

Prove that there exists a unique function $v = v(i, j)$ satisfying properties (a) and (b). [Start doing the particular case $k = m = 4$, in which there are 12 boundary squares in (a) and 4 interior squares in (b); use simple linear algebra. After that, try to find an argument to prove the same for a given couple (k, m) of integers].

Problem 3. Let Ω be the annulus $\{x \in \mathbb{R}^2 : 1 < |x| < 2\}$ of $\mathbb{R}^2$. For $x \in \Omega$ let $u(x)$ be the probability of exiting from Ω moving randomly, and beginning from x, when the boundary $\{|x| = 1\}$ is open and $\{|x| = 2\}$ is closed. To solve this problem, use that $u = 1$ on $\{|x| = 1\}$ and $u = 0$ on $\{|x| = 2\}$, and that the solution is unique.

(i) Without doing any computation, try to guess if $u(3/2, 0)$ is smaller or larger than $1/2$. Can you give a proof of this?

(ii) Compute explicitly the function $u = u(x)$, $x \in \Omega$.

Problem 4. (a) In our random walk game in $\Omega \subset \mathbb{R}^2$, now all the boundary is closed and what we care about is measuring for "how long" the random trajectories starting from a given point $x \in \Omega$ stay inside Ω before hitting the boundary $\partial\Omega$. Discretize the problem to be able to give a measure of such "how long", and then deduce which is the final PDE for the limiting function $v = v(x)$. We call $v(x)$ the (expected) exit time from Ω when starting from x.

(b) Let $\Omega = \{x \in \mathbb{R}^2 : 1 < |x| < 2\}$. Consider the points of Ω in which the exit time from Ω is maximum. (b1) Without doing any calculation, could you say from which of the two boundaries of the annulus they are nearer? (b2) Find which are exactly these points of maximum.