Pseudo-rigid bodies: A geometric Lagrangian approach

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Abstract

The pseudo-rigid body model is viewed in the context of continuum mechanics and elasticity theory. A Lagrangian reduction, based on variational principles, is developed for both anisotropic and isotropic pseudo-rigid bodies. For isotropic Lagrangians the reduced equations of motion for the pseudo-rigid body are a system of two (coupled) Lax equations on $\mathfrak{so}(3) \times \mathfrak{so}(3)$ and a second order differential equation on the set of diagonal matrices with positive determinant. Several examples of pseudo-rigid bodies such as stretching bodies, spinning gas could and Riemann ellipsoids are presented.

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1 Introduction

A pseudo-rigid body (or affine-rigid body) is a deformable body with motion characterized by an orientation preserving linear map. This is not the most general type of deformable body for which its motion is, in general, an orientation preserving diffeomorphism. However the pseudo-rigid body model is a fairly rich framework for dealing with problems involving rigid motions and deformations. The assumption of linearity on the allowed motions makes the theory of pseudo-rigid bodies very attractive since it implies that their dynamics is governed by a system of ordinary differential equations with a finite number of degrees of freedom and hence more tractable than the initial-boundary problems for deformable media.

The mechanics of pseudo-rigid bodies can be viewed as a generalization of rigid body classical Mechanics and at the same time as a particular case of continuum Mechanics. If one thinks that the rigid body is an idealization, since no body is truly rigid, then the pseudo-rigid body is the right framework for the study of bodies that behave as an almost rigid body. Examples are the long time motion of a satellite where small deformations become important in its orbital control and the Earth motion (the Earth is not rigid).

The pseudo-rigid body model has been applied in a series of studies such as: Cohen and Muncaster [5, Chapter 6] use the perturbation analysis of the motion of a pseudo-rigid body in a neighbourhood of a rigid state to study gyroscopic motions due to the flexibility of the body; Dyson [6] uses this model for a spinning gas could motion; several classical masters, Poincaré, Cartan, Liapounoff, Roche, Darwin and Jeans, have used the most well known example of a pseudo-rigid body, the motion of self gravitating fluid masses, for the study of the birth of planets and double stars (see Chandrasekar [3] for an overview and historical references).

The basic geometry of a pseudo-rigid body is given by the polar decomposition of a configuration which provides an effective tool for analysing the interplay between orientation and deformation. Other geometric properties are furnished by the so-called constitutive assumptions for different types of body materials. These assumptions are expressed in terms of invariant functions under certain Lie groups. Thus, the geometric richness of the pseudo-rigid body model makes it quite appropriated for the use of a geometric approach and techniques such as group reduction, momentum maps and energy-momentum method (Lewis and Simo [7] study the nonlinear stability of some particular examples of pseudo-rigid bodies using the energy momentum method).

The main aim of this work is to obtain the equations of motion for a pseudo-rigid body by exploring its geometric structure. For this we use a Lagrangian formulation and group reduction based on variational principles. One of the main advantages of the Lagrangian formulation is that once the reduced motion equations have been solved the reconstruction of the dynamics in the original space is easy by comparison with the Hamiltonian analog (see remark on page 15). Also, in some examples of pseudo-rigid bodies (see last example) the Legendre transform is not invertible and so the Lagrangian formalism is more appropriated than the Hamiltonian one.

Although some new results are presented, namely the reduction results, our main purpose was to give an introduction to the subject and at the same time to show the importance of a geometrical approach. We hope that the selection of topics and the perspective used will motivate future research directions and interests.

2 Pseudo-rigid body symmetries

A reference body $\mathcal{B}$ is the closure of an open set in $\mathbb{R}^3$ with a sufficiently smooth boundary. A configuration of $\mathcal{B}$ is a smooth enough, orientation preserving, injective (except possibly on the boundary of $\mathcal{B}$) map $\phi : \mathcal{B} \to \mathbb{R}^3$. A motion of $\mathcal{B}$ is a time-dependent family of configurations $x = \phi(X, t)$.

A *pseudo-rigid body configuration* is a (time-dependent) orientation preserving linear map. So, for a pseudo-rigid body the so-called deformation gradient of a configuration, $\nabla\phi$, coincides with the configuration.

Identify a motion $F(t)$ of a reference body with (an orientation preserving) 3×3 matrix, i.e an element of the linear group $GL(3)$ with positive determinant. That is the configuration space for the pseudo-rigid body by $GL^+(3)$.

We will adopt a Lagrangian formulation and use of the invariance of the Lagrangian function under actions of subgroups of the configuration space. The equations of motion of a pseudo-rigid body will be obtained by group reduction. This reduction is based on variational principles behind the Newton's fundamental law of force balance $\mathbf{F} = m\mathbf{a}$. Let us recall Hamilton's variational principle.

We choose the velocity phase space to be the tangent bundle of the configuration space, $TGL^+(3)$, and the dynamics determined by a Lagrangian function $\mathcal{L} : TGL^+(3) \to \mathbb{R}$. Coordinates of the configuration space and of its tangent bundle will be denoted respectively by F and $(F, \dot{F})$. The variational principle of Hamilton states that the variation of the action is stationary at a solution:

$$\delta\mathfrak{S} = \delta \int_a^b \mathcal{L}(F, \dot{F}, t)\, dt = 0 \tag{1}$$

The application of this principle is done by choosing curves $F(t)$ on the configuration space joining two fixed points on it over the fixed time intervale $[a, b]$, and regarding the action $\mathfrak{S}$ as function of these curves. The Hamilton's principle states that the action has a critical point at a solution in the space of curves with some fixed endpoints. Hamilton's principle is equivalent to the Euler-Lagrange equations:

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{F}} = \frac{\partial\mathcal{L}}{\partial F}$$

(the precise meaning for the partial derivatives will be given later on).

As we will see the geometric structure of a pseudo-rigid body configuration allows us to reduce the Euler-lagrange equations to Euler type equations on Lie subalgebras of the Lie algebra of $GL^+(3)$. This is a very convenient approach since the symmetry properties of the Lagrangian function reflect in many cases the symmetries of the constitutive assumptions which characterize different types of materials.

2.1 The polar and bipolar decomposition

In order to better understand the geometric structure of the pseudo-rigid body configuration space let us recall the so-called polar decomposition.

Any real invertible matrix F, $n \times n$, can be factored in a unique fashion as

$$F = SU \quad \text{or} \quad F = VR, \tag{2}$$

where S, R are orthogonal matrices and U, V are positive definite symmetric matrices. One has $U^2 = F^T F$, $V^2 = FF^T$ and $V^2 = SU^2S^T$.

The orientation of the pseudo-rigid body is characterized by the rotation matrix L and the effects of the deformation by the matrix U.

The two matrices U^2 and V^2, and their eigenvalues, play a key role in the form of general response functions. The matrix $U^2 = F^T F$ (respectively $V^2 = FF^T$) is called the *right Cauchy-Green strain tensor* (respectively the *left Cauchy-Green strain tensor*). For reference see, for instance, Marsden and Hughes [10] and Ciarlet [4].

Furthermore, the positive definite symmetric matrices U and V are orthogonally diagonalized and so we can conclude that there are matrices $L, R \in SO(n)$ such that $F = LDR$, where D is a diagonal matrix whose diagonal entries are the square roots of the singular values of F. The decomposition $F = LUR$ is known as bipolar decomposition.

The question of existence of a continuous bipolar decomposition for a motion is always true if we consider analytic motions and even non analytic if the singular values of F are two by two distinct. However there are C^∞ motions for which there is no continuous bipolar decomposition (see Kato [9]).

Hereafter whenever we use the bipolar decomposition of a pseudo-rigid body motion we assume the existence of a smooth enough bipolar decomposition.

2.2 Constitutive assumptions and Symmetries

The references for this section are mainly Gurtin [8], Marsden and Hughes [10] and Ciarlet [4].

Continuum mechanics is based on some basic principles such as Cauchy principle, being the media dynamics governed by initial boundary-value problems derived from momentum balance laws. Momentum balance laws are common to all type of bodies and independent of the materials they are made of. However, physical experience shows that bodies with the same shape and size, for instance one made of iron and other of liquid, subject to the same system of forces behave differently. In order to encompass different types of material behaviour it is necessary to impose additional hypotheses known as constitutive assumptions. Some other hypotheses are very common in continuum mechanics and elasticity theory, such as frame indifference and isotropy. These can be easily translated in terms of symmetry properties of the stress tensors entering in Lagrangian function which governs the dynamical process.

Forces in continuum mechanics are described by body forces distributed over the volume and forces distributed over oriented surfaces. These forces can be measured per unit of volume, respectively of area, either in the reference body $\mathcal{B}$ or in the current configuration $\phi(\mathcal{B})$. The stress measuring the surface force in the reference configuration is known as the *first Piola Kirchhoff stress*, represented here by P, being Pn the force exerted across an oriented surface, with n the unit normal to the surface. The stress measuring the force per unit of area in the current configuration, T, is called *Cauchy stress.* This is related to P by $T = \frac{1}{\det F} PF^T$, where F is the deformation gradient of the configuration. Note that although the Cauchy stress is symmetric there is no reason for the Piola Kirchhoff stress to be.

Several types of constitutive assumptions are common in continuum mechanics. Namely, Gurtin [8] distinguishes the following types:

- Constraints on the possible deformations: For instance rigid motions or incompressibility. The first one is the basis of rigid body mechanics while the second is usual for some liquids and even for some gases. The incompressibility assumption is given by the constancy of the volume which means also that the determinant of the deformation gradient is equal to 1.

- Assumptions on the form of the stress: A very common assumption is that the Cauchy stress has the form $T = pI$, where p is a real vector field depending on the motion, and I the identity matrix. That is the Cauchy stress is a pressure. For instance an ideal fluid is an incompressible body for which T is a pressure (in this case even the mass density function is constant).

- Constitutive equations relating the stress to the motion: For instance an elastic fluid is a body where the pressure is a scalar function of the density ρ. For instance an ideal gas is an elastic fluid with pressure $p = \lambda\rho^\gamma$ where $\lambda > 0$ and $\gamma > 1$ are constants.

Usually in Mechanics some other properties of the motion are assumed such as frame-indifference, homogeneity and isotropy. The axiom of *frame indifference* is taken for granted in mechanics since it is too obvious. Roughly, it asserts that the measurement, made by two observers placed in different positions, of the forces necessary to produce some motion should be the same. Indeed the motion differs only by a change in observer. In terms of the stress tensors, T and P, this means that they satisfy the following symmetry property:

$$T(X, QF) = QT(X, F)Q^T \qquad P(X, QF) = QP(X, F), \quad Q \in SO(3) \qquad (3)$$

where F is the deformation gradient. For instance ideal fluids are frame indifferent since T is a pressure and so $T(F) = -pI = -pQ^TQ = T(QF)$.

Consider now *elastic bodies* that is bodies for which the stress tensors T and P depend only on the deformation gradient F and not on its derivatives. In particular for a pseudo-rigid body this means

$$T(x,t) = \hat{T}(F(t), X) \qquad P(X,t) = \hat{P}(F(t), X) \quad (x = F(t)X) \tag{4}$$

In what follows we suppress the point X in the notation.

The polar decomposition gives that the stress tensors for an elastic body are determined by their restriction to the set of positive definite symmetric matrices:

Lemma 2.1. *The Cauchy and the first Piola-Kirchhoff stress tensors for an elastic pseudo-rigid body are determined by their restriction to the set of positive definite symmetric matrices*

$$\hat{T}(F) = L\hat{T}(U)L^T \qquad \hat{P}(F) = L\hat{P}(U) \quad L \in SO(3)$$

where U is the tensor corresponding to the polar decomposition $F = LU$.

The stress tensors for isotropic materials can be written in terms of the so-called principal invariants of the Cauchy-Green strain tensors. Indeed the Cayley-Hamilton theorem gives that any tensor satisfies its own characteristic equation. That is, if A is for instance a 3×3 matrix, we have

$$\det(A - \lambda I) = -\lambda^3 + I_1(A)\lambda^2 - I_2(A)\lambda + I_3(A),$$

and so

$$A^3 - I_1(A)A^2 + I_2(A)A - I_3(A) = 0,$$

where I_1, I_2, I_3 are the so-called principal invariants of A. The principal invariants of a 3×3 matrix A are

$$I_1(A) = \mathrm{tr}(A), \quad I_2(A) = \tfrac{1}{2}\left[(\mathrm{tr}A)^2 - \mathrm{tr}(A^2)\right], \quad I_3(A) = \det(A). \tag{5}$$

The Cauchy-Green tensors are symmetric and so their principal invariants are completely determined by their spectra being their principal invariants the symmetric functions of the eigenvalues of these tensors.

Definition 2.2. *The symmetry group of a point $X \in \mathcal{B}$ is defined as*

$$G_X = \left\{ Q \in GL^+(n) : \hat{T}(X,F) = \hat{T}(X,FQ) \right\}.$$

A point X is said to be isotropic if $G_X \subseteq SO(3)$ and anisotropic otherwise.

A material is said to be *homogeneous* if either both the density function and the stresses T and P are independent of points of the reference body. Note that for homogeneous materials the symmetry group of the material is also independent of X.

Let us mention that it is possible to classify the materials in terms of the symmetry group of the material. For instance, solids have symmetry group $SO(3)$, fluids $SL(3)$ and crystals a group lying in between $SO(3)$ and $SL(3)$. (see Marsden and Hughes [10, pg. 10] and references therein).

In this paper we are mainly concerned with body motions subject to conservative forces. That is, bodies for which there exists a stored energy function W such that the Piola Kirchhoff stress is equal to the derivative of W with respect to the deformation gradient F. In Elasticity these type of bodies are known as hyperelastic. For practical purposes the most interesting stored energy functions are the ones that are frame indifferent and isotropic at each point of the reference body. In this case, if $F = LDR \in GL^+(3)$ and W is the stored energy function then

$$\begin{aligned} W(X, F^T F) &= W(X, (R^T D^2 R)) \\ &= W(X, D^2 R) && \text{by frame indifference} \\ &= W(X, D^2) && \text{by isotropy} \\ &= W(X, I_1(D^2), I_2(D^2), I_3(D^2)) \end{aligned}$$

where I_1, I_2 and I_3 are the principal invariants of D^2. Namely, for $D = \operatorname{diag}\big(a_1, a_2, a_3\big)$, we have

$$I_1(D^2) = \operatorname{tr}(D^2) = a_1^2 + a_2^2 + a_3^2$$

$$I_2(D^2) = \tfrac{1}{2}\left[\big(\operatorname{tr}(D^2)\big)^2 - \operatorname{tr}(D^4)\right] = a_1^2 a_2^2 + a_2^2 a_3^2 + a_1^2 a_3^2 = \det(D^2)\operatorname{tr}((D^2)^{-1})$$

$$I_3(D^2) = \det(D^2) = a_1^2 a_2^2 a_3^2.$$

If W is a differentiable function then

$$\begin{aligned} \frac{\partial W}{\partial D} &= \frac{\partial W}{\partial I_1}\frac{\partial I_1}{\partial D} + \frac{\partial W}{\partial I_2}\frac{\partial I_2}{\partial D} + \frac{\partial W}{\partial I_3}\frac{\partial I_3}{\partial D} \\ &= 2\left[\frac{\partial W}{\partial I_1} D + \frac{\partial W}{\partial I_2}\left(I_1 D - D^3\right) + \frac{\partial W}{\partial I_3} I_3 D^{-1}\right]. \end{aligned}$$

For the value of the derivatives $\frac{\partial I_1}{\partial D}, \frac{\partial I_2}{\partial D}, \frac{\partial I_3}{\partial D}$ see for instance [4, 10].

Let us now mention some of the most common materials used in the applications with this kind of stored energy functions. The reference for all of the following materials is Ciarlet [4, section 4] and references therein.

(a) *Mooney-Rivlin materials*

For compressible Mooney-Rivlin materials we have

$$W(D) = aI_1(D^2) + bI_2(D^2) + \Gamma(\det(D)),$$

$$a, b > 0 \text{ and } \Gamma(\delta) = c\delta^2 - d\operatorname{Log}(\delta),\ c, d > 0,$$

and for incompressible Mooney-Rivlin materials the same function without the term $\Gamma(\det(D))$.

(b) *St Venant-Kirchhoff materials*

$$W(D) = -\left(\tfrac{3\lambda+2\mu}{4}\right) I_1(D^2) + \left(\tfrac{\lambda+2\mu}{8}\right) I_1(D^4) + \left(\tfrac{\lambda}{4}\right) I_2(D^2) + \left(\tfrac{9\lambda+6\mu}{8}\right),$$

where $\lambda, \mu > 0$ are the so-called Lamé constants which are determined by the experiments (see table 3.8.4 of Ciarlet [4] for the values of these constants for several materials like steel, iron, glass, lead, rubber and so on).

(c) *Neo-Hookean materials*

For compressible neo-Hookean materials we have

$$W(D) = aI_1(D^2) + \Gamma(\det(D)), \qquad a > 0$$

and for incompressible neo-Hookean materials the same function without the term $\Gamma(\det(D))$.

(d) *Hadammard-Green materials*

$$W(D) = \tfrac{\alpha}{2} I_1(D^2) + \tfrac{\beta}{2}\left(I_1^2(D^2) - I_1(D^4)\right) + \Gamma(\det(D)), \qquad \alpha, \beta > 0.$$

(e) *Ogden's materials*

$$W(D) = \textstyle\sum_{i=1}^{M} \alpha_i I_1\left(D^{\gamma_i}\right) + \sum_{i=1}^{N} \beta_i I_2\left(D^{\delta_i}\right) + \Gamma(\det(D)),$$

$$\alpha_i > 0, \gamma_i, \delta_i \geq 1 \text{ and } \Gamma(\delta) \to +\infty \text{ as } \delta \to 0^+.$$

3 Anisotropic pseudo-rigid body Lagrangians

In this and following section we present a reduction for pseudo-rigid bodies which can be viewed as an extension of the so-called Euler-Poincaré reduction (see Marsden and Ratiu [11] for Euler-Poincaré reduction). The relation between the reduction presented here and the so-called Lagrange-Poincaré reduction (see Cendra, Marsden and Ratiu [1]) is out of the scope of this paper but it will be interesting to be done. For references and an overview of the several types of reduction either for Lagrangian and Hamiltonian systems with symmetry see Cendra, Marsden and Ratiu [2].

Consider the dynamics of a pseudo-rigid body given by a Lagrangian function $\mathcal{L}$ invariant under the left $SO(3)$ action on $TGL^+(3)$. That is

$$\mathcal{L}(F, \dot{F}) = \mathcal{L}(QF, Q\dot{F}) \quad \text{for} \quad Q \in SO(3).$$

Proposition 3.1. *A $SO(3)$-invariant Lagrangian function $\mathcal{L} : TGL^+(3) \to \mathbb{R}$ induces a function l on $\mathfrak{so}(3) \times TS_+$ where $\mathfrak{so}(3)$ is the Lie algebra of $SO(3)$ and TS_+ is the tangent bundle of the set of positive definite symmetric matrices of order* 3.

The variational principle for $\mathfrak{L}$ reduces to the variational principle

$$\delta \int l(\xi(t), U(t), \dot{U}(t))\, dt = 0$$

for variations $\delta\xi$ verifying

$$\delta\xi = \dot{\zeta} + [\xi, \zeta]$$

where ζ and δU vanish at fixed endpoints. Furthermore the Euler-Lagrange equations for l are (coupled) system of Euler type equations on $\mathfrak{so}(3) \times TS_+$ given by:

$$\begin{aligned} &\tfrac{d}{dt}\,\tfrac{\delta l}{\delta \xi} = \mathrm{ad}^*_\xi\,\tfrac{\delta l}{\delta \xi} \\ &\tfrac{d}{dt}\,\tfrac{\delta l}{\delta \dot{U}} = \tfrac{\delta l}{\delta U} \end{aligned} \tag{6}$$

where $(U, \dot{U}) \in TS_+$ and $\xi \in \mathfrak{so}(3)$.

Proof. A motion $F(t)$ decomposes $F(t) = L(t)U(t) \in TGL^+(3)$ where L and U are respectively an orthogonal matrix and a positive definite symmetric matrix. As $\mathfrak{L}$ is left $SO(3)$-invariant then

$$\begin{aligned} \mathfrak{L}(F(t), \dot{F}(t)) \quad &= \mathfrak{L}(LU, \dot{L}U + L\dot{U}) = \mathfrak{L}(U, \xi U + \dot{U}) \\ &= l(\xi(t), U(t), \dot{U}(t)), \end{aligned}$$

where $\xi(t) = L^T(t)\dot{L}(t) \in \mathfrak{so}(3)$ and $(U(t), \dot{U}(t)) \in TS_+$.

Let $L(\epsilon, t)$ be a curve on $SO(3)$ and $\delta L = \frac{d}{d\epsilon}\, L(\epsilon, t)\Big|_{\epsilon\,=\,0}$. So a short computation gives

$$\delta\xi = \frac{d}{d\epsilon}\, L^{-1}(\epsilon, t)\dot{L}(\epsilon, t)\Big|_{\epsilon\,=\,0} = \dot{\zeta} + [\xi, \zeta],$$

where $\zeta = L^{-1}\delta L$.

Taking variations ζ and δU vanishing at $t = a$ and $t = b$ with $\delta\xi = \dot{\zeta} + \mathrm{ad}_\xi \zeta$ the variational principle for l is

$$\begin{aligned} \delta \int_a^b l(\xi, U, \dot{U}) dt \quad &= \int_a^b \langle \tfrac{\delta l}{\delta \xi}\,,\, \delta\xi\rangle\, dt + \int_a^b \langle \tfrac{\delta l}{\delta U}\,,\, \delta U\rangle\, dt + \int_a^b \langle \tfrac{\delta l}{\delta \dot{U}}\,,\, \delta\dot{U}\rangle\, dt \\ &= \int_a^b \langle \tfrac{\delta l}{\delta \xi}\,,\, \dot{\zeta} + \xi\zeta - \zeta\xi\rangle\, dt + \int_a^b \langle \tfrac{\delta l}{\delta U} - \tfrac{d}{dt}\tfrac{\delta l}{\delta \dot{U}}\,,\, \delta U\rangle\, dt \\ &= \int_a^b \langle \mathrm{ad}^*_\xi \tfrac{\delta l}{\delta \xi} - \tfrac{d}{dt}\tfrac{\delta l}{\delta \xi}\,,\, \zeta\rangle\, dt + \int_a^b \langle \tfrac{\delta l}{\delta U} - \tfrac{d}{dt}\tfrac{\delta l}{\delta \dot{U}}\,,\, \delta U\rangle\, dt = 0 \end{aligned}$$

where integration by parts and the vanishing conditions for ζ and δU have been used. As the above equality holds for all ζ and δU satisfying the referred conditions, so equations (6) hold. □

In order to highlight the above reduction and for further reference we start by applying it to the case when the pseudo-rigid body is a rigid body. In this case the configuration space is $SO(3)$ and the above reduction is just the reduction by the left action of a Lie group G on its tangent space TG. The reduced equations are equations on the Lie algebra of G. This reduction is known as Euler-Poincaré reduction and the reduced equations are called the Euler-Poincaré equations (see Marsden and Ratiu [11]).

3.1 Free rigid body

A rigid body can be viewed as a particular example of a pseudo-rigid body since a configuration for the rigid body is $F = LU = L$, i.e $U(t)$ is the identity for all t. So the configuration space is just the special orthogonal group $SO(3)$. The Lagrangian function for a free rigid body is just given by the kinetic energy:

$$\mathcal{L}(F, \dot{F}) = \frac{1}{2} \int_{\mathcal{B}} \rho(X) \|\dot{F}X\|^2 \, dV,$$

where $\rho(X)$ is the mass density function, $F \in SO(3)$ and dV denotes the volume element.

As F is an orthogonal matrix then $\dot{F} = FF^T\dot{F} = F\xi$ for the skew symmetric matrix $\xi = F^T\dot{F}$ (i.e $\xi \in \mathfrak{so}(3)$). The kinetic energy is left $SO(3)$-invariant since

$$\|\dot{F}X\|^2 = \|F\xi X\|^2 = X^T\xi^T\xi X = \|\xi X\|^2.$$

Thus $\mathcal{L}$ induces a function l on $\mathfrak{so}(3)$ given by

$$\mathcal{L}(F, \dot{F}) = \frac{1}{2} \int_{\mathcal{B}} \rho(X) \, \|\dot{F}X\|^2 \, dV = \frac{1}{2} \int_{\mathcal{B}} \rho(X) \, \|\xi X\|^2 \, dV = l(\xi).$$

Furthermore, as $\xi^T\xi$ is a symmetric matrix an easy computation shows that

$$l(\xi) \quad = \tfrac{1}{2} \int_{\mathcal{B}} \rho(X) \, \|\xi X\|^2 \, dV = \tfrac{1}{2} \int_{\mathcal{B}} \rho(X) \, X^T\xi^T\xi X \, dV = \tfrac{1}{2} \mathrm{tr} \left(\xi^T\xi\mathbb{I}\right),$$

where $\mathbb{I}$ is the symmetric (inertia) tensor:

$$\mathbb{I} = \frac{1}{2} \int_{\mathcal{B}} \rho(X) \, XX^T \, dV. \tag{7}$$

Elementary properties of the trace of a matrix gives

$$\begin{aligned} \tfrac{\delta l}{\delta \xi} \cdot \delta\xi \quad &= \tfrac{1}{2} \mathrm{tr} \left((\delta\xi)^T\xi\mathbb{I} + \xi(\delta\xi)^T\mathbb{I}\right) = \tfrac{1}{2} \mathrm{tr} \left[(\xi\mathbb{I} + \mathbb{I}\xi) \, (\delta\xi)^T\right] \\ &= \tfrac{1}{2} \langle\langle \xi\mathbb{I} + \mathbb{I}\xi \, , \, \delta\xi \rangle\rangle = \langle\langle \mathrm{sk}(\xi\mathbb{I}) \, , \, \delta\xi \rangle\rangle. \end{aligned} \tag{8}$$

That is $\frac{\delta l}{\delta \xi} = \mathrm{sk}(\xi\mathbb{I})$, where sk denotes the skew symmetric part of a matrix ($\mathrm{sk}(C) = \frac{C - C^T}{2}$).

Moreover for any matrices A, B, C we have

$$\begin{aligned} \langle\langle \mathrm{ad}_A^* B \, , \, C \rangle\rangle \quad &= \langle\langle B \, , \, \mathrm{ad}_A C \rangle\rangle = \langle\langle B \, , \, AC - CA \rangle\rangle = \mathrm{tr} \left(BC^TA^T - BA^TC^T\right) \\ &= \mathrm{tr} \left[\left(A^TB - BA^T\right) C^T\right) = \langle\langle A^TB - BA^T \, , \, C \rangle\rangle \\ &= \langle\langle \mathrm{ad}_{A^T} B \, , \, C \rangle\rangle. \end{aligned}$$

The reduced equations are then:

$$\frac{d}{dt}\frac{\delta l}{\delta \xi} = \mathrm{ad}^*_\xi \frac{\delta l}{\delta \xi} \Longleftrightarrow \frac{d}{dt}\left(\mathrm{sk}(\xi \mathbb{I})\right) = \mathrm{ad}_{\xi^T}\left(\mathrm{sk}(\xi\mathbb{I})\right) = -\,\mathrm{ad}_\xi\left(\mathrm{sk}(\xi\mathbb{I})\right). \tag{9}$$

Let us show that the above equations are exactly the Euler equations for the free rigid body. Consider the isomorphism of the Lie algebras $(\mathbb{R}^3, \times)$ and $(\mathfrak{so}(3), [\,,\,])$ given by

$$\mathbb{R}^3 \ni \omega = (w_1, w_2, w_3) \mapsto \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix} = \xi \in \mathfrak{so}(3). \tag{10}$$

Straightforward calculations give, for $\xi, \zeta \in \mathfrak{so}(3)$ and $\mathbb{I}$ a symmetric matrix,

$$\mathrm{sk}(\xi\mathbb{I}) = \frac{1}{2}\left(\mathbb{I}\xi + \xi\mathbb{I}\right) = \frac{1}{2}\left(\mathrm{tr}(\mathbb{I})I - \mathbb{I}\right)\omega = \tilde{\mathbb{I}}\,\omega, \tag{11}$$

where ω is the $\mathbb{R}^3$ vector identification of ξ.

Then $\mathrm{sk}(\xi\mathbb{I}) = \tilde{\mathbb{I}}\omega$ where $\tilde{\mathbb{I}}$ is the moment of inertia tensor defined by (11) and $-\mathrm{ad}_\xi\left(\mathrm{sk}(\xi\mathbb{I})\right) = -\omega \times \tilde{\mathbb{I}}\,\omega = \tilde{\mathbb{I}}\,\omega \times \omega$. That is, the reduced equations (9) are equal to

$$\frac{d}{dt}\left(\tilde{\mathbb{I}}\,\omega\right) = \tilde{\mathbb{I}}\,\omega \times \omega. \tag{12}$$

$\Pi = \tilde{\mathbb{I}}\,\omega \in \mathfrak{so}^*(3)$ is the angular momentum in the body frame. Since $\tilde{\mathbb{I}}$ is a positive definite matrix then it is invertible and (12) is equivalent to $\dot{\Pi} = \Pi \times \tilde{\mathbb{I}}^{-1}\Pi$ which are precisely the Euler equations for the rigid body. The Euler equations are Hamiltonian equations on the dual $so^*(3)$, with respect to the Poisson structure

$$\left\{F, H\right\}(\Pi) = -\Pi \cdot \left(\nabla F(\Pi) \times \nabla H(\Pi)\right).$$

See Marsden and Ratiu [11] for details.

$\diamond$

In order to get more practical implications for the application of proposition 3.1 we assume that the body is elastic and that the forces involved are conservative. These assumptions are equivalent to consider Lagrangian functions of the form

$$\begin{aligned} &\mathfrak{L}(F, \dot{F}) = K(\dot{F}) - \sigma(F) - W(F) \\ &\sigma(F) = \textstyle\int_{\mathcal{B}} \bar{\sigma}(X, F)\, dX, \qquad W(F) = \textstyle\int_{\mathcal{B}} \bar{W}(X, F)\, dX, \end{aligned} \tag{13}$$

where W is the internal energy function, σ the stored energy function corresponding to surface forces and $\bar{W}$ and $\bar{\sigma}$ the corresponding pointwise energy functions.

3.2 Homogeneous stretching pseudo-rigid body

Consider a frame indifferent (anisotropic) pseudo-rigid body, for instance in the form of a cube, in absence of body forces with configuration is given by

$$FX = \begin{pmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_2 \end{pmatrix}, \tag{14}$$

where $\alpha = 2\tan k$, being k the angle between the undeformed and deformed e_2-axis.

If we compute the polar decomposition, $F = LU$, we get for L and U respectively

$$L = \begin{pmatrix} \cos k & \sin k & 0 \\ -\sin k & \cos k & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad U = \begin{pmatrix} \cos k & \sin k & 0 \\ \sin k & \frac{1+\sin^2 k}{\cos k} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

being L a rotation of an angle k about the third axis.

Consider the Lagrangian function given by

$$\mathcal{L}(F, \dot{F}) = K(\dot{F}) - \sigma(F) = K(\xi U + \dot{U}) - \sigma(LU) = K(\xi U + \dot{U}) - \sigma(U),$$

where $\xi = L^T \dot{L}$ and the last equality follows from the assumption of frame indifference. So the Lagrangian function $\mathcal{L}$ induces a lagrangian function l on $\mathfrak{so}(3) \times TS_+$ given by

$$l(\xi, U, \dot{U}) = K(\xi U + \dot{U}) - \sigma(U).$$

In particular if the body is homogeneous then $\bar{\sigma}$ does not depend on X and l is given by

$$\mathcal{L}(F, \dot{F}) = l(\xi, U, \dot{U}) = \frac{1}{2}\int_{\mathcal{B}} \rho \|(\xi U + \dot{U})X\|^2 \, dX - \int_{\mathcal{B}} \bar{\sigma}(U)\, dX. \tag{15}$$

The proposition 3.1 applied to this type of material gives the following result:

Corollary 3.2. *The equations of motion of a homogeneous, frame indifferent and anisotropic 3-dimensional stretching pseudo-rigid body in absence of body forces are:*

$$\frac{d}{dt}\Big(\mathrm{sk}\,(\xi U \mathbb{I} U + \dot{U}\mathbb{I}U)\Big) = -\Big[\xi\,,\, \mathrm{sk}\,(\xi U \mathbb{I} U + \dot{U}\mathbb{I}U)\Big]$$

$$\frac{d}{dt}\mathrm{sy}\,(\dot{U}\mathbb{I}) = \mathrm{sy}\,(\xi^T \xi U \mathbb{I} + \mathbb{I}\dot{U}\xi) - \frac{\partial \sigma}{\partial U},$$

where $\mathbb{I}$ *is the inertia tensor (7),* $\xi \in \mathfrak{so}(3)$, $\mathrm{sk}(C) = \frac{C - C^T}{2}$ *and* $\mathrm{sy}(C) = \frac{C + C^T}{2}$.

Proof. First note that $\frac{\delta l}{\delta \xi} = \frac{\delta K}{\delta \xi}$, $\frac{\delta l}{\delta \dot{U}} = \frac{\delta K}{\delta \dot{U}}$ and $\frac{\delta l}{\delta U} = \frac{\delta K}{\delta U} - \frac{\partial \sigma}{\partial U}$ where $\frac{\partial \sigma}{\delta U} = \left[\frac{\partial \sigma}{\delta U_{ij}} \right]$ for $U = [U_{ij}]$. The kinetic energy is given by:

$$
\begin{aligned}
K(\xi, U, \dot{U}) &= \tfrac{1}{2} \int_{\mathcal{B}} \rho \|(\xi U + \dot{U})X\|^2 \, dX \\
&= \tfrac{1}{2} \int_{\mathcal{B}} \rho \left(\|\xi U X\|^2 + \|\dot{U}X\|^2 + 2(\xi U X) \cdot (\dot{U}X) \right) dX \\
&= \tfrac{1}{2} \left(\operatorname{tr}(U\xi^T \xi U \mathbb{I}) + \operatorname{tr}(\dot{U}\dot{U}\mathbb{I}) + \operatorname{tr}(\dot{U}\xi U \mathbb{I} + U \xi^T \dot{U} \mathbb{I}) \right),
\end{aligned}
$$

where $\mathbb{I}$ is the inertia tensor and a straightforward computation gives the expression of the last term for the non symmetric matrix $\dot{U}\xi U$. Differentiating K and using the pairing $\langle\langle\, ,\, \rangle\rangle$ given in (8), we get:

$$
\langle \tfrac{\delta K}{\delta \dot{U}} \, , \, \delta \dot{U} \rangle = \langle\langle \operatorname{sy}(\dot{U}\mathbb{I}) \, , \, \delta \dot{U} \rangle\rangle, \qquad \langle \tfrac{\delta K}{\delta \xi} \, , \, \delta \xi \rangle = \langle\langle \operatorname{sk}(\xi U \mathbb{I} U + \dot{U}\mathbb{I}U) \, , \, \delta \xi \rangle\rangle
$$

$$
\langle \tfrac{\delta K}{\delta U} \, , \, \delta U \rangle = \langle\langle \operatorname{sy}(\xi^T \xi U \mathbb{I} + \mathbb{I}\dot{U}\xi) \, , \, \delta U \rangle\rangle.
$$

So using proposition 3.1 the result follows. $\square$

Since U is known, the above equations can be written in terms of α and $\dot{\alpha}$. We leave this as an exercise. $\diamond$

4 Isotropic Pseudo-rigid body Lagrangians

Isotropic materials are the most interesting for practical purposes since their constitutive functions only depend on the principal invariants of the Cauchy-Green tensors. That is, the constitutive functions for frame indifferent isotropic materials are characterized by their invariance under both left and right $SO(3)$ actions on $TGL^+(3)$. One may think to apply the same type of reduction by considering Lagrangian functions with the same type of invariance of the constitutive functions for isotropic materials. However the kinetic energy entering in the Lagrangian is not in general invariant under the right $SO(3)$ action. Indeed if $F = LDR$, with L and R time-dependent orthogonal matrices and D a diagonal matrix, the kinetic energy is given by:

$$
\begin{aligned}
K(\dot{F}) &= \tfrac{1}{2} \int_{\mathcal{B}} \rho(X) \, \|\dot{F}X\|^2 \, dX = \tfrac{1}{2} \int_{\mathcal{B}} \rho(X) \left\| \left(\xi D + \dot{D} + D\eta \right) RX \right\|^2 dX \\
&= \tfrac{1}{2} \int_{R(\mathcal{B})} \rho(R^T Y) \left\| \left(\xi D + \dot{D} + D\eta \right) Y \right\|^2 dY.
\end{aligned}
$$

So, for a Lagrangian function to be left and right $SO(3)$ invariant, i.e

$$
\mathcal{L}(F, \dot{F}) = \mathcal{L}(QFS, Q\dot{F}S) \quad \text{for} \quad Q, S \in SO(3), \tag{16}
$$

it is necessary to impose several additional conditions. For instance if the mass density function is frame-indifferent (or even constant as in the case of a homogeneous body) and the reference body spherically symmetric, then the kinetic energy is invariant under the right $SO(3)$ action. One can also ask for reference bodies having the symmetry of a subgroup of $SO(3)$.

Having these considerations in mind, hereafter we restrict to classes of motion for which the Lagrangian function verifies (16).

Proposition 4.1. *A Lagrangian function $\mathcal{L}$ for an isotropic pseudo-rigid body, satisfying (16), induces a Lagrangian function $l : \mathfrak{so}(3) \times \mathfrak{so}(3) \times T\mathcal{D}_+$ where $\mathfrak{so}(3)$ is the Lie algebra of $SO(3)$ and $T\mathcal{D}_+$ is the tangent bundle of the group of the positive definite diagonal matrices. The variational principle for $\mathcal{L}$ reduces to the variational principle*

$$\delta \int l(\xi, \eta, D, \dot{D})\, dt = 0$$

for variations

$$\delta\xi = \dot{\zeta} + \mathrm{ad}_\xi \zeta, \qquad \delta\eta = \dot{\alpha} - \mathrm{ad}_\eta \alpha,$$

where $\delta D, \zeta$ and α vanish at fixed endpoints. Furthermore the variational principle for l is equivalent to the following system:

$$\begin{aligned} &\tfrac{d}{dt}\,\tfrac{\delta l}{\delta \xi} = \mathrm{ad}^*_\xi\, \tfrac{\delta l}{\delta \xi} \\ &\tfrac{d}{dt}\,\tfrac{\delta l}{\delta \eta} = -\mathrm{ad}^*_\eta\, \tfrac{\delta l}{\delta \eta} \\ &\tfrac{d}{dt}\,\tfrac{\delta l}{\delta \dot{D}} = \tfrac{\delta l}{\delta D} \end{aligned} \tag{17}$$

The proof of this proposition follows exactly as for proposition 3.1. Here we just present a sketch aiming to clarify some points.

Proof. Let $F(t) = L(t)D(t)R(t) \in GL^+(3)$ where L, R are (time dependent) orthogonal matrices and D is a diagonal matrix. So

$$\dot{F} = \dot{L}DR + L\dot{D}R + LD\dot{R} = L\xi DR + L\dot{D}R + LD\eta R,$$

where $\xi = L^{-1}\dot{L} \in so(3)$ and $\eta = \dot{R}R^{-1} \in so(3)$. Then the invariance property (16) of $\mathcal{L}$ gives

$$\begin{aligned} \mathcal{L}(F, \dot{F}) &= \mathcal{L}\left(LDR, L\xi DR + L\dot{D}R + LD\eta R\right) \\ &= \mathcal{L}\left(D, \xi D + \dot{D} + D\eta\right) = l(\xi, \eta, D, \dot{D}) \end{aligned}$$

Note that the sign difference in the form of $\delta\xi$ and $\delta\eta$ is due to the following:

$$\delta\eta = \frac{d}{d\epsilon}\Big|_{\epsilon=0}\left(\dot{R}(t,\epsilon)R^{-1}(t,\epsilon)\right) = \frac{d^2R}{dtd\epsilon}\Big|_{\epsilon=0} R^{-1} - \dot{R}R^{-1}(\delta R)R^{-1}.$$

So, taking $\alpha = (\delta R)R^{-1}$ and differentiating with respect to t we get

$$\dot{\alpha} = \frac{d^2 R}{dt d\epsilon}\Big|_{\epsilon\,=\,0} R^{-1} - (\delta R)R^{-1}\dot{R}R^{-1},$$

and so $\delta\eta - \dot{\alpha} = -\eta\alpha + \alpha\eta = [\alpha, \eta] = -\mathrm{ad}_\eta \alpha$.

The computation of $\delta \int_a^b l(\xi, \eta, D, \dot{D})dt = 0$ follows exactly as in the proof of proposition 3.1 using integration by parts and the vanishing conditions on the fixed endpoints. □

Before proceeding with examples let us remark how we reconstruct the dynamics from the reduced equations. The reconstruction centers on the equations

$$\xi(t) = L^T(t)\dot{L}(t) \quad \text{and} \quad \eta(t) = \dot{R}(t)R^T(t), \tag{18}$$

and can be phrased as: For $\xi(t)$, $\eta(t)$ and $D(t)$ solutions of (17) with initial conditions ξ_0,η_0 and D_0 respectively, solve the equations (18) for $L(t)$ and $R(t)$ with the given initial conditions. Then the solution of the Euler-Lagrange equations for $\mathcal{L}$ with initial condition F_0 is $F(t) = L(t)D(t)R(t)$.

4.1 Free isotropic pseudo-rigid body

For future use and reference we give in the next proposition the form of the reduced equations (17) for a 3-dimensional pseudo-rigid body for which its Lagrangian is just the kinetic energy and the density function is frame indifferent. We restrict also to a class of bodies satisfying $R\mathcal{B} = \mathcal{B}$ for any R belonging to $SO(3)$ or to a subgroup of it. We call this pseudo-rigid body a free isotropic pseudo- rigid body. We will not present the proof for the equations of motion since it involves lengthy computations and do not present more difficulty than the computations for the rigid body example and the proof of corollary 3.2.

Let $\rho(QX) = \rho(X)$ for any $Q \in SO(3)$ and $F = LDR$ with L, R orthogonal matrices and D diagonal. Consider the Lagrangian function to be the kinetic energy. Thus

$$\begin{aligned} \mathcal{L}(F, \dot{F}) &= \tfrac{1}{2} \int_{\mathcal{B}} \rho(X) \, \|\dot{F}X\|^2 \, dX = \tfrac{1}{2} \int_{\mathcal{B}} \rho(X) \left\| \left(\xi D + \dot{D} + D\eta\right) X \right\|^2 dX \\ &= l(\xi, \eta, D, \dot{D}), \end{aligned}$$

where $\xi = L^T\dot{L}$ and $\eta = \dot{R}R^T$.

Proposition 4.2. *The equations (17) for the free isotropic pseudo-rigid body are verified with*

$$\begin{aligned} &\tfrac{\delta l}{\delta \xi} = \mathrm{sk}\left(\xi D\mathbb{I}D + D\eta\mathbb{I}D + \dot{D}\mathbb{I}D\right) \qquad \tfrac{\delta l}{\delta \eta} = \mathrm{sk}\left(D^2\eta\mathbb{I} + D\xi D\mathbb{I} + D\dot{D}\mathbb{I}\right) \\ &\tfrac{\delta l}{\delta \dot{D}} = \mathrm{sy}_d\left(\dot{D}\mathbb{I} + \xi D\mathbb{I} + D\eta\mathbb{I}\right) \\ &\tfrac{\delta l}{\delta D} = \mathrm{sy}_d\left(\xi^T\xi D\mathbb{I} + D\eta\mathbb{I}\eta^T + \mathbb{I}\eta^T D\xi + \eta\mathbb{I}D\xi^T + \eta\mathbb{I}\dot{D} + \mathbb{I}\dot{D}\xi\right) \end{aligned} \tag{19}$$

where $\mathbb{I}$ *is the inertia tensor* $\mathbb{I} = \int_{\mathcal{B}} \rho(X) X X^T dY$ *and* $\mathrm{sy}_d(C)$ *represents the diagonal matrix with diagonal equal to the diagonal of* $\mathrm{sy}(C)$*.*

The inertia tensor $\mathbb{I}$ is symmetric and so one can choose a reference frame in which it is diagonal. Furthermore if the body $\mathcal{B}$ is symmetric relative to the origin of this frame then $\mathbb{I}$ is a multiple of the identity, say $\mathbb{I} = \mu I$. In this case we can obtain simpler expression for the reduced equations. We will call this type of body a symmetric isotropic pseudo-rigid body.

Corollary 4.3. *The equations of motion in* $\mathfrak{so}(3) \times \mathfrak{so}(3) \times T\mathcal{D}_+$ *for a symmetric isotropic free pseudo-rigid body, is given by a second order equation on the diagonal matrices,*

$$\ddot{D} = -\,\mathrm{sy}_d\left(\xi^2 D + D\eta^2 + 2\eta D \xi\right) \tag{20}$$

and the following (coupled) system on $\mathfrak{so}(3) \times \mathfrak{so}(3)$ *(identified with* $\mathbb{R}^3 \times \mathbb{R}^3$*):*

$$\begin{aligned} &\tfrac{d}{dt}(M\omega + K\lambda) = (M\omega + K\lambda) \times \omega \\ &\tfrac{d}{dt}(K\omega + M\lambda) = -(K\omega + M\lambda) \times \lambda \end{aligned} \tag{21}$$

where ω, λ *are the vector identifications of* ξ *and* η *respectively, and* M, K *are the following* 3×3 *diagonal matrices:*

$$M = \frac{1}{2}\left[\mathrm{tr}(D^2) I - D^2\right], \qquad K = (\det D)\, D^{-1}.$$

Moreover, if the shape matrix is $D = \mathrm{diag}(a_1, a_2, a_3)$*, then*

$$M = \frac{1}{2}\,\mathrm{diag}\bigg(a_2^2 + a_3^2, a_1^2 + a_3^2, a_1^2 + a_2^2\bigg) \quad \text{and} \quad K = \mathrm{diag}\bigg(a_2 a_3, a_1 a_3, a_1 a_2\bigg).$$

Proof. Let us deduce first the equations (21). Since $\mathbb{I}, D$ and $\dot{D}$ are diagonal they commute and so $\mathrm{sk}\big(\dot{D}\mathbb{I}D\big) = 0 = \mathrm{sk}\big(D\dot{D}\mathbb{I}\big)$. Then expressions (19) reduce to:

$$\tfrac{\delta l}{\delta \xi} = \mathrm{sk}\left(\xi D^2 \mathbb{I} + D\eta D \mathbb{I}\right) \qquad \tfrac{\delta l}{\delta \eta} = \mathrm{sk}\left(D^2 \eta \mathbb{I} + D\xi D\mathbb{I}\right).$$

As by hypothesis, the pseudo rigid body is symmetric, i.e $\mathbb{I} = \mu I$, then μ cancels in the equations (17) for $\frac{\delta l}{\delta \xi}$ and $\frac{\delta l}{\delta \eta}$.

Now, as D^2 is a symmetric matrix then by (11) we have:

$$\begin{aligned} \mathrm{sk}\left(\xi D^2\right) &= \tfrac{1}{2}\left(\xi D^2 - D^2 \xi^T\right) = \tfrac{1}{2}\left(\xi D^2 + D^2 \xi\right) \\ &= \tfrac{1}{2}\left[\mathrm{tr}(D^2) I - D^2\right] \omega = M\omega \end{aligned} \tag{22}$$

and similarly $\mathrm{sk}\left(D^2 \eta\right) = \frac{1}{2}\left[\mathrm{tr}(D^2) I - D^2\right] \lambda = M\lambda$.

Also, as for any invertible matrix A the isomorphism (10) gives

$$A\lambda \mapsto (\det A)\, A^{-T} \eta A^{-1}, \tag{23}$$

then

$$\mathrm{sk}\,(D\eta D) \quad = \tfrac{1}{2}\,(D\eta D + D\eta D) = D\eta D = (\det D)\, D^{-1}\lambda = K\lambda.$$

So $\frac{d}{dt}\frac{\delta l}{\delta \xi} = \mathrm{ad}^*_\xi \frac{\delta l}{\delta \xi} = -\mathrm{ad}_\xi \frac{\delta l}{\delta \xi}$ is equivalent to

$$\frac{d}{dt}(M\omega + K\lambda) = -\omega \times (M\omega + K\lambda).$$

Similarly for the other expression. A straightforward computation gives the form of K and M.

For the second order equation in D note that when $\mathbb{I} = \mu I$ it commutes with D and $\dot{D}$ and $\mathrm{sy}_d(\eta\mathbb{I}\dot{D} + \mathbb{I}\dot{D}\eta) = 0$. So the expressions for $\frac{\delta l}{\delta D}$ and $\frac{\delta l}{\delta \dot{D}}$ reduce to

$$\frac{\delta l}{\delta \dot{D}} = \mathrm{sy}_d\big(\dot{D}\mathbb{I}\big) = \mu\,\mathrm{sy}_d(\dot{D}) \qquad \frac{\delta l}{\delta D} = \mu\,\mathrm{sy}_d\big((\xi^T\xi D + D\eta\eta^T - 2\eta D\xi\big).$$

Furthermore as $\mathrm{sy}_d(\dot{D}) = \dot{D}$ then the result follows.

□

In Roberts and Sousa-Dias [12] it is proved that $M\omega - K\lambda$ and $-K\omega + M\lambda$ are respectively the angular momentum and circulation for the pseudo-rigid body. Also, in the referred work, it is proved that these vectors are the components of the momentum map for the lift of $SO(3) \times SO(3)$ action, $(L, R) \cdot Q \mapsto LQR$, to the cotangent bundle $T^*GL^+(3)$.

Due to the similarity of the Lax equation obtained for the free rigid body and for the pseudo-rigid, one may ask if the Lax equations for the pseudo-rigid body are equivalent, via the Legendre transform, to Hamiltonian equations in the dual, $\mathfrak{so}^*(3) \times \mathfrak{so}^*(3)$, with respect to some Poisson structure. One can easily see that in the case of the pseudo-rigid body the Legendre transform is not always invertible since the 6×6 matrix $\begin{bmatrix} M & -K \\ -K & M \end{bmatrix}$ is only invertible if the shape matrix D have all entries two by two distinct. This means that in this case the Lagrangian and the Hamiltonian formalisms are not equivalent.

In the following sections we only consider isotropic and frame indifferent materials subject to conservative forces.

We can conclude that if the reference body is in the conditions of corollary 4.3 and its motion modeled by the Lagrangian function of the form

$$\mathfrak{L}(F, \dot{F}) = K(\dot{F}) - W(F^T F) = l(\xi, D, \dot{D}) = K(\xi D + \dot{D} + D\eta) - W(D^2),$$

then the equations of motion are given by equations (21) and the equation (20) is substituted by

$$\ddot{D} = -\,\mathrm{sy}_d\left(\xi^2 D + \eta^2 D + 2\eta D\xi\right) - \frac{\partial W}{\partial D}.$$

4.2 Stretching Mooney-Rivlin pseudo-rigid body

Let us apply corollary 4.3 to example of page 12 of a stretching body made of an isotropic material. Consider that the body is symmetric relative to the origin of a reference frame in which $\mathbb{I}$ is diagonal, that is $\mathbb{I} = \mu I$.

By the bipolar decomposition of F we know that the principal invariants of D^2 are the principal invariants of FF^T. If F is as in (14) then

$$FF^T = \begin{pmatrix} 1+\alpha^2 & \alpha & 0 \\ \alpha & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\det(FF^T - \lambda I) = -\lambda^3 + (3+\alpha^2)\lambda^2 - (3+\alpha^2)\lambda + 1.$$

So, the principal invariants of D^2 are $I_1(D^2) = 3+\alpha^2, I_2(D^2) = 3+\alpha^2$ and $I_3(D^2) = 1$.

Suppose now that the cube is a Mooney-Rivlin material, that is the stored energy function is given by

$$W(D) = a(3+\alpha^2) + b(3+\alpha^2) + c \qquad a, b, c > 0.$$

The matrices M and K are respectively $M = \frac{1}{2}\left[(3+\alpha^2)I - D^2\right]$, $K = D^{-1}$. The equations of motion are

$$\ddot{D} = -\mathrm{sy}_d(\xi^2 D + \eta^2 D + 2\eta D\xi) - \tfrac{2}{\mu}\left(a + b(3+\alpha^3) - bD^2\right) D$$

$$\tfrac{d}{dt}\left(((3+\alpha^2)I - D^2)\omega + D^{-1}\lambda\right) = \left((3+\alpha^2)I - D^2)\omega + D^{-1}\lambda\right) \times \omega$$

$$\tfrac{d}{dt}\left(D^{-1}\omega + ((3+\alpha^2)I - D^2)\lambda\right) = -\left(D^{-1}\omega + (3+\alpha^2)I - D^2)\lambda\right) \times \lambda.$$

Note that the second order equation in D is equivalent to a second order equation in α which can be easily computed. For note that D is the diagonal matrix having entries the square roots of the principal invariants of D^2.

◇

Let us end with two more elaborated examples of application of proposition 4.1. The first one, the motion of a spinning gas cloud was proposed by Dyson [6] and the second is the so-called Dirichlet problem for self-gravitating fluid masses or Riemann ellipsoids (see Chandrasekhar [3]).

4.3 Spinning (pseudo-rigid) gas cloud

The properties of the gas is described by the density ρ, the pressure P, the temperature T and the internal energy U per gram. All these functions are considered to depend on time and on points of the reference body. Consider that the configuration of the reference body $\mathcal{B}$ is a linear function, that is the gas cloud is a pseudo-rigid body. Furthermore, if we consider that the total mass is given by a standard Gaussian density distribution f, that is the total mass is $\int_{\mathcal{B}} f dX$, then the density function is frame indifferent and the kinetic energy is $SO(3)$-right invariant.

Assuming that the gas is isothermal, that is the temperature does not depend on points X of the reference body, and the internal energy function is independent on the density (which does not happen for all gases but is true for instance for perfect gases where $U = nkT$ with n and k constants), then the internal energy is only a function of the determinant of the configuration F (see Dyson [6] for details). Note that the pressure has also a known expression as function of the density and temperature.

Under these conditions the motion of the gas cloud is modeled by a Lagrangian function $\mathcal{L}(F, \dot{F}) = K(\dot{F}) - U(F)$ where K is both left and right $SO(3)$-invariant and U is isotropic since it depends only on the determinant of D.

So, $\mathcal{L}$ induces a Lagrangian function on $\mathfrak{so}(3)\times\mathfrak{so}(3)\times T\mathcal{D}_+$ and the equations of motion are given by proposition 4.1, that is

$$\begin{aligned}
&\mathbb{I}\ddot{D} = -\mathrm{sy}_d\left(\xi\xi D\mathbb{I} + D\eta\mathbb{I}\eta + \mathbb{I}\eta D\xi + \eta\mathbb{I}D\xi\eta\xi D\right) - \det(D)\,\frac{dU}{dI_3(D)}D^{-1}\\
&\tfrac{d}{dt}(\tilde{M}\omega + \tilde{K}\lambda) = (\tilde{M}\omega + \tilde{K}\lambda)\times\omega\\
&\tfrac{d}{dt}(\tilde{K}\omega + \tilde{M}\lambda) = -(\tilde{K}\omega + \tilde{M}\lambda)\times\lambda,
\end{aligned}$$

where $\lambda, \omega \in \mathbb{R}^3$ isomorphic respectively to $\eta, \xi \in \mathfrak{so}(3)$ under the isomorphism (10) and $\tilde{M}, \tilde{K}$ matrices depending on D and $\mathbb{I}$ (see proposition 4.4 for their expressions).

The first equation gives the rate of expansion of the gas cloud under the influence of the pressure force $\frac{\partial U}{\partial D}$, and the two other equations give six equations in the angular velocity and circulation involving only inertial properties of the gas cloud.

One can compare the above equations with the second order equation given by Dyson. It can be proved that the off-diagonal terms of Dyson's equation are exactly our six equations for the angular velocity and circulation and the diagonal terms are equivalent to our second order equation in D.

$\diamond$

4.4 Self Gravitating Fluid Masses (Riemann Ellipsoids)

The study of ellipsoidal figures of equilibrium for a homogeneous self gravitating fluid mass has a long story which can be traced back Newton's time in his studies of the shape of the Earth. This problem have attracted the attention of many Classical masters such as Riemann, Lagrange, Legendre and Poincaré (only to refer a few). Riemann was the first one who gave conditions for the existence of ellipsoidal figures of equilibria (equilibria with constant angular velocity and circulation) and this is the reason for the problem to bear his name nowadays. For a very complete reference on the subject see Chandrasekhar [3].

The first formulation as a pseudo-rigid body problem is due to Dirichlet who asked: "Under what conditions can one have a configuration which at every instant, has an ellipsoidal figure and in which motion in an inertial frame, is a linear function of the coordinates?".

In order to apply the theory we developed in this work, some questions are natural to raise. The first one is if the gravitational potential entering in the Lagrangian function is or not isotropic. A second question is when the kinetic energy it is right $SO(3)$-invariant. The answer to the first question is immediate from the form of the gravitational potential given below, and the answer to the second one has yet been answered positively not only for spherical reference bodies but also for motions relative to which the reference body is invariant under rotations characterized by the circulation.

The motion $F(t)$ of a self-gravitating homogeneous (pseudo-rigid) fluid mass which has always an ellipsoidal shape is modeled by a (constrained) Lagrangian of the form

$$\mathcal{L}(F,\dot{F}) = K(\dot{F}) - p(\det(\mathrm{F})) - V(F)$$

subject to the condition of $p=0$ on the boundary,$\partial\mathcal{B}$, of the ellipsoid. V is the gravitational potential and p is the pressure.

Hereafter we restrict ourselves to the class of motions for which the kinetic energy is right $SO(3)$-invariant. For $F = LDR$ and $D = \mathrm{diag}\big(a_1, a_2, a_3\big)$ the gravitational potential is

$$\begin{aligned} V(D) &= -2\pi G\rho\left(\det D\right)\int_0^\infty \frac{du}{\sqrt{(a_1^2+u)(a_2^2+u)(a_3^2+u)}} \\ &= -2\pi G\rho\left(\det D\right)\int_0^\infty \frac{du}{\sqrt{u^3+I_1(D^2)u^2+I_2(D^2)u+I_3(D^2)}} \end{aligned}$$

Choosing a reference frame in which the inertia tensor $\mathbb{I}$ is diagonal, the proposition 4.2 give the following proposition.

Proposition 4.4. *The equations of motion for a configuration $F = LDR$, of a homogeneous self-gravitating fluid mass for which the kinetic energy is right invariant for a subgroup of $SO(3)$ are given by*

$$\begin{aligned} \tfrac{d}{dt}\big(\tilde{M}\omega + \tilde{K}\lambda\big) &= \big(\tilde{M}\omega + \tilde{K}\lambda\big)\times\omega \\ \tfrac{d}{dt}\big(\tilde{M}\omega + \tilde{K}\lambda\big) &= -\big(\tilde{K}\omega + \tilde{M}\lambda\big)\times\lambda \\ \mathbb{I}\ddot{D} &= \mathrm{sy}_d\left(\xi\xi^T D\mathbb{I} + D\eta\mathbb{I}\eta^T + \mathbb{I}\eta^T D\xi + \eta\mathbb{I}D\xi^T\right) \\ &\qquad -p\det(D)\,D^{-1} - 2\left[\tfrac{\partial V}{\partial I_1}D + \tfrac{\partial V}{\partial I_2}\big(I_1 D - D^3\big) + \tfrac{\partial V}{\partial I_3}I_3 D^{-1}\right] \end{aligned}$$

subject to the conditions of $p = 0$ *on* $\partial\mathcal{B}$ *and the incompressibility condition given by the constancy of* $\det D$.

Furthermore the matrices $\tilde{M}$ *and* $\tilde{K}$ *are given by*

$$\tilde{M} = \frac{1}{2}\big[\mathrm{tr}(D^2\mathbb{I}) - D^2\mathbb{I}\big], \qquad \tilde{K} = \frac{1}{2}\det(D\mathbb{I})\big[\mathrm{tr}(\mathbb{I}^{-1})I - \mathbb{I}^{-1}\big]\mathbb{I}^{-1}D^{-1},$$

where ξ, η, ω *and* λ *are as in corollary 4.3.*

We can compare the above equations with that of Chandrasekhar's [3, pg. 73] and see that it corresponds exactly the same system. Also we can prove that the derivative $\frac{\partial V}{\partial D} = -2\pi G\mathcal{U}D$ presented in Chandrasekhar's book coincide with our expression.

The form of the matrices $\tilde{M}, \tilde{K}$ are obtained applying the same properties used in the computation of the matrices M, K of corollary 4.3. Namely as D and $\mathbb{I}$ commute

$$\mathrm{sk}(\xi D^2\mathbb{I}) = \frac{1}{2}\left(\xi D^2\mathbb{I} + D^2\mathbb{I}\xi\right) = \frac{1}{2}\left[\mathrm{tr}(\mathrm{D}^2\mathbb{I})\,\mathrm{I} - \mathrm{D}^2\mathbb{I}\right]\omega = \tilde{M}\,\omega,$$

where the last equality follows from (22). For $\tilde{K}$ we apply both (22) and (23):

$$\begin{aligned} \mathrm{sk}(D\eta D\mathbb{I}) &= \tfrac{1}{2}\left(D\eta D\mathbb{I} + \mathbb{I}D\eta D\right) = \tfrac{1}{2}\left[\mathbb{I}^{-1}\left(D\mathbb{I}\,\eta\, D\mathbb{I}\right) + \left(D\mathbb{I}\,\eta\, D\mathbb{I}\right)\mathbb{I}^{-1}\right] \\ &= \tfrac{1}{2}\det(D\mathbb{I})\big[\mathrm{tr}(\mathbb{I}^{-1})I - \mathbb{I}^{-1}\big]\mathbb{I}^{-1}D^{-1}\lambda. \end{aligned}$$

In Roberts and Sousa-Dias [12] Riemann's theorem is obtained for relative equilibria of a Hamiltonian system, where the Hamilton function is invariant under both the left and right $SO(3)$ actions on the configuration space, by studying the symmetry properties of the momentum map. A comparison with the expressions obtained there and $\tilde{M}\omega - \tilde{K}\lambda$ and $-\tilde{K}\omega + \tilde{M}\lambda$ allow us to say that conditions for the existence of equilibria of the two first equations of proposition 4.4 give the so-called Riemann's theorem.

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