

# The group of symplectic birational maps of the plane and the dynamics of a family of 4D maps

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## Abstract

We approach the dynamics of a family of birational maps  $\varphi_k$  in dimension 4, using Poisson geometry tools, namely the properties of the restrictions of the maps  $\varphi_k$  and their fourth iterate  $\varphi_k^{(4)}$  to the symplectic leaves of an appropriate Poisson manifold  $(\mathbb{R}_+^4, P)$ . These restricted maps are shown to belong to a group of symplectic birational maps of the plane which is isomorphic to the semidirect product  $SL(2, \mathbb{Z}) \ltimes \mathbb{R}^2$ . The study of these restricted maps leads to the conclusion that there are three different types of dynamical behaviour for  $\varphi_k$  characterized by the parameter values  $k = 1$ ,  $k = 2$  and  $k \geq 3$ .

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## 1 Introduction

This is a companion paper to our works [2, 5] on the dynamics of maps arising in the context of the theory of cluster algebras [6] through the notion of mutation-periodic quivers [9] (a.k.a. cluster maps). We study the main geometric features underpinning the dynamics of a family of (cluster) maps  $\varphi_k$  in dimension 4, depending on a positive integer parameter  $k$ . Although most of the dynamical behaviour of these maps is presented in our unpublished work [4], here we approach their dynamics under a different point of view, aiming at keeping the paper as self contained as possible and at the same time highlighting the main geometric aspects relevant to the dynamics.

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We consider the family of maps defined in  $\mathbb{R}_+^4$  by

$$\varphi_k(x_1, x_2, x_3, x_4) = \left( x_3, x_4, \frac{x_2^k + x_3^k}{x_1}, \frac{x_1^k x_4^k + (x_2^k + x_3^k)^k}{x_1^k x_2} \right), \quad k \in \mathbb{Z}_+ \quad (1)$$

This family is associated to a 4-node quiver which is mutation-periodic of period 2 and is a particular case of the quiver presented in [2, Figure 1, Example 4]. By definition of the maps associated to mutation periodic quivers, the maps  $\varphi_k$  are birational, that is, rational maps with rational inverse.

We refer the reader interested in mutation-periodic quivers and studies of maps associated to mutation-periodic quivers of period 1 to [9, 8, 7] and to [11, 10] for general aspects of cluster algebras and applications.

We approach the study of the dynamics of the maps  $\varphi_k$  by realizing that they are maps preserving a Poisson structure  $P$  of *log-canonical* type. This structure is regular on  $\mathbb{R}_+^4$ , and the leaves of the respective symplectic foliation of  $\mathbb{R}_+^4$  are semi-algebraic sets of dimension 2. All these symplectic leaves are invariant under the fourth iterate  $\varphi_k^{(4)}$  of the maps  $\varphi_k$  with one leaf being invariant under  $\varphi_k$ . The periodic points of the maps  $\varphi_k$  are then obtained by studying the restrictions of  $\varphi_k$  and  $\varphi_k^{(4)}$  to the (invariant) symplectic leaves. This study provides the full description of the periodic points of the family of maps (1) and enables us to conclude that there are three different types of dynamical behaviour according to the parameter values  $k = 1$ ,  $k = 2$  and  $k \geq 3$ . The identification of the periodic points of  $\varphi_k$  in the cases  $k = 1$  and  $k \geq 3$  is described in Theorem 3 and Theorem 4 respectively, and for  $k = 2$  it can be found in [3, Theorem 3]. In particular, we show that: (a)  $\varphi_1$  is globally 12-periodic, with a unique fixed point and 2-dimensional semi-algebraic sets of points with minimal periods 4 and 6; (b)  $\varphi_2$  has no periodic points; (c) if  $k \geq 3$ ,  $\varphi_k$  has a unique fixed point and a 2-dimensional semi-algebraic set of points of minimal period 4.

The structure of the paper is as follows. The first section recalls that each map  $\varphi_k$  is a Poisson map with respect to a Poisson structure  $P$  of rank 2. We also show that the respective symplectic foliation of  $\mathbb{R}_+^4$  is invariant under the fourth iterate of  $\varphi_k$  (Theorem 1). The following section is devoted to the study of the restrictions of  $\varphi_k$  and of  $\varphi_k^{(4)}$  to the appropriate (invariant) symplectic leaves of the referred foliation. It is shown that these restrictions are symplectic birational maps of the plane belonging to a group,  $\Gamma$ , which is isomorphic to the semidirect product  $SL(2, \mathbb{Z}) \ltimes \mathbb{R}^2$ . We also find normal forms for the maps of this group up to conjugation in  $GL(2, \mathbb{Z}) \ltimes \mathbb{R}^2$ . The final section is devoted to the study of the periodic points of the maps  $\varphi_k$ . This is accomplished by using results of the previous section concerning the group  $\Gamma$  to obtain the periodic points of the restricted maps, from which we determine the periodic points of the maps  $\varphi_k$ .

## 2 Reduction to globally-periodic symplectic maps

Although each map  $\varphi_k$  of the family (1) is defined for  $x_1x_2 \neq 0$ , throughout the paper we consider its domain of definition to be  $\mathbb{R}_+^4$  which guarantees that any iterate of these maps is well defined.

As follows from Example 4 in [2] (with  $r = s = k$  and  $t = 0$ ) each map  $\varphi_k$  is a Poisson map with respect to the Poisson structure

$$P = \sum_{1 \leq i < j \leq 4} c_{ij} x_i x_j \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \quad (2)$$

where the matrix  $C = [c_{ij}]$  is the skew-symmetric matrix

$$C = \begin{bmatrix} 0 & k & k & k^2 \\ -k & 0 & 0 & k \\ -k & 0 & 0 & k \\ -k^2 & -k & -k & 0 \end{bmatrix}. \quad (3)$$

That is, for each  $k \in \mathbb{Z}_+$  we have  $(\varphi_k)_* P = P$ , where  $(\varphi_k)_*$  denotes the push-forward by  $\varphi_k$ . The Poisson structure  $P$  is known as a log-canonical Poisson structure since it is constant in logarithmic coordinates.

It can easily be checked that  $[c_{ij} x_i x_j]_{i,j=1,\dots,4}$  has null determinant and consequently, in  $\mathbb{R}_+^4$ , the Poisson tensor has constant rank equal to 2, meaning that  $P$  is a regular (degenerate) Poisson structure.

Each map  $\varphi_k$  is a birational Poisson map and so by Theorem 5.1 in [2] there is a submersion  $\Pi_k$  and a map  $\hat{\varphi}_k$  defined on  $\mathbb{R}_+^2$  such that

$$\Pi_k \circ \varphi_k = \hat{\varphi}_k \circ \Pi_k. \quad (4)$$

That is, one has the commutativity of the following diagram

$$\begin{array}{ccc} \mathbb{R}_+^4 & \xrightarrow{\Pi_k} & \mathbb{R}_+^2 \\ \varphi_k \downarrow & & \downarrow \hat{\varphi}_k \\ \mathbb{R}_+^4 & \xrightarrow{\Pi_k} & \mathbb{R}_+^2 \end{array}$$

Moreover, the submersion  $\Pi_k$  sends  $\mathbb{R}_+^4$  onto a maximal set of independent Casimirs. Such a set can be easily obtained from a basis of the kernel of the matrix  $C$  in (3). Indeed, if  $\mathbf{v} = (v_1, v_2, v_3, v_4) \in \text{Ker } C$  then  $\mathbf{x}^{\mathbf{v}} := x_1^{v_1} x_2^{v_2} x_3^{v_3} x_4^{v_4}$  is a Casimir (see Lemma 5.2 in [2]). Thus, considering  $\text{Ker } C = \langle \mathbf{v}_1, \mathbf{v}_2 \rangle$  with

$$\mathbf{v}_1 = (1, -k, 0, 1), \quad \mathbf{v}_2 = (0, -1, 1, 0),$$

we take for a maximal set of Casimirs,  $\{x, y\}$ , the rational functions  $x = \mathbf{x}^{\mathbf{v}_1}$  and  $y = \mathbf{x}^{\mathbf{v}_2}$ , that is,

$$x = \frac{x_1 x_4}{x_2^k}, \quad y = \frac{x_3}{x_2}. \quad (5)$$

The submersions  $\Pi_k$  are then given by

$$\Pi_k(x_1, x_2, x_3, x_4) = \left( \frac{x_1 x_4}{x_2^k}, \frac{x_3}{x_2} \right) = (x, y). \quad (6)$$

The components of the maps  $\widehat{\varphi}_k$  are obtained by computing  $x \circ \varphi_k$  and  $y \circ \varphi_k$  as functions of  $x$  and  $y$ . This computation gives

$$\widehat{\varphi}_k(x, y) = \left( \frac{y(x^k + (1 + y^k)^k)}{x^k}, \frac{1 + y^k}{x} \right). \quad (7)$$

**Remark 1.** We note that each Casimir is invariant under a scaling action of the multiplicative group  $\mathbb{R}_+^2$  on  $\mathbb{R}_+^4$  with weights defined by the components of vectors forming a basis of the image of the matrix  $C$  in (3). Namely, taking  $\text{Im } C = \langle \mathbf{u}_1, \mathbf{u}_2 \rangle$  with  $\mathbf{u}_1 = (0, k, k, k^2)$  and  $\mathbf{u}_2 = (-k, 0, 0, k)$ , this scaling action is defined by

$$(\lambda_1, \lambda_2) \cdot (x_1, x_2, x_3, x_4) = (\lambda_2^{-k} x_1, \lambda_1^k x_2, \lambda_1^k x_3, \lambda_1^{k^2} \lambda_2^k x_4), \quad (\lambda_1, \lambda_2) \in (\mathbb{R}_+)^2.$$

However, the maps  $\varphi_k$  are not invariant (neither equivariant) under this scaling action, which leaves us outside the usual Poisson reduction setting.

Another remark is worth mentioning.

**Remark 2.** As the Poisson structure  $P$  in (2) is regular in  $\mathbb{R}_+^4$  all its (2-dimensional) symplectic leaves are the common level sets of a maximal set of independent Casimirs. However, it can be shown that the maps  $\varphi_k$  do not preserve this symplectic foliation of  $\mathbb{R}_+^4$ , and therefore none of these maps is the discrete analogue of a Hamiltonian flow.

Taking into account the above remark one might be tempted to consider that the Poisson structure is of no relevance to the study of the dynamics of the family of maps  $\varphi_k$ . However, as we will show in Theorem 1, the fourth iterate of each map  $\varphi_k$  does preserve the symplectic foliation of  $\mathbb{R}_+^4$ . Hence, like in the continuous setting, the restriction of these maps to the symplectic leaves can be used to study the dynamics.

**Theorem 1.** *The fourth iterate of each map  $\varphi_k$  in (1) preserves the symplectic foliation of  $(\mathbb{R}_+^4, P)$  with  $P$  the Poisson structure in (2). That is,*

$$\varphi_k^{(4)} \left( S_{(p,q)}^k \right) \subseteq S_{(p,q)}^{(4)}$$

where  $S_{(p,q)}^k$  is a symplectic leaf. In particular, the Casimirs  $\tilde{x} = \frac{x_2^k + x_3^k}{x_1 x_4}$  and  $y = \frac{x_3}{x_2}$  are first integrals of  $\varphi_k^{(4)}$ , i.e.,

$$\tilde{x} \circ \varphi_k^{(4)} = \tilde{x}, \quad y \circ \varphi_k^{(4)} = y.$$

This theorem is a different formulation of Proposition 4 in [5]. In order to prove it we first show the following proposition.

**Proposition 1.** Let  $\widehat{\varphi}_k$  be the maps in (7) defined in  $\mathbb{R}_+^2$ . Then,

- (i) each map  $\widehat{\varphi}_k$  is globally 4-periodic, i.e.  $\widehat{\varphi}_k^{(4)} = Id$ ;
- (ii) each map  $\widehat{\varphi}_k$  is symplectic with respect to the symplectic form

$$\omega = \frac{1}{xy} dx \wedge dy.$$

*Proof.* (i) Consider the homeomorphisms

$$h_k(x, y) = \left( y, \frac{1 + y^k}{x} \right), \quad k \in \mathbb{Z}_+.$$

A simple computation shows that

$$h_k \circ \widehat{\varphi}_k = \psi \circ h_k \tag{8}$$

with

$$\psi(x, y) = \left( y, \frac{1}{x} \right). \tag{9}$$

The parameter independent map  $\psi$  is globally 4-periodic (i.e.  $\psi^{(4)} = Id$ ), and from (8) it follows

$$h_k \circ \widehat{\varphi}_k^{(4)} = \psi^{(4)} \circ h_k \iff \widehat{\varphi}_k^{(4)} = Id,$$

where the above equivalence comes from the global periodicity of  $\psi$  and the fact that  $h_k$  is a homeomorphism.

- (ii) Straightforward computations show that the maps  $h_k$  and  $\psi$  preserve  $\omega$ , that is the pullback of  $\omega$  by these maps is  $\omega$ :

$$h_k^* \omega = \omega, \quad \psi^* \omega = \omega.$$

Thus

$$(\widehat{\varphi}_k)^* \omega = (h_k^{-1} \circ \psi \circ h_k)^* \omega = h_k^* \circ \psi^* \circ (h_k^{-1})^* \omega = \omega.$$

□

The symplectic leaves of  $(\mathbb{R}_+^4, P)$  are 2-dimensional subsets of  $\mathbb{R}_+^4$  (since the rank of  $P$  is 2) defined by the common level set of two independent Casimirs of  $P$ . These leaves could be defined as the fibres of the submersion  $\Pi_k$  in (6) but due to the previous proposition it is more convenient to consider them to be the fibres of

$$\pi_k(\mathbf{x}) = h_k \circ \Pi_k(\mathbf{x}) = \left( y, \frac{1 + y^k}{x} \right) = \left( \frac{x_3}{x_2}, \frac{x_2^k + x_3^k}{x_1 x_4} \right). \tag{10}$$

We note that, since  $x$  and  $y$  are Casimirs of  $P$  then  $\frac{1+y^k}{x}$  is also a Casimir and so the components of  $\pi_k$  form a maximal set of independent Casimirs. Thus, the symplectic leaves are given by

$$\begin{aligned} S_{(p,q)}^k &= \{\mathbf{x} \in \mathbb{R}_+^4 : \pi_k(\mathbf{x}) = (p, q)\} \\ &= \{\mathbf{x} \in \mathbb{R}_+^4 : x_3 = px_2, qx_1x_4 = (1+p^k)x_2^k\} \end{aligned} \quad (11)$$

with  $(p, q) \in \mathbb{R}_+^2$ .

*Proof of Theorem 1.* From (4) and (8) one has

$$\Pi_k \circ \varphi_k = \widehat{\varphi}_k \circ \Pi_k \iff h_k \circ \Pi_k \circ \varphi_k = \psi \circ h_k \circ \Pi_k \iff \pi_k \circ \varphi_k = \psi \circ \pi_k \quad (12)$$

with  $\pi_k$  the map (10) and  $\psi$  as in (9). The last equivalence implies

$$\pi_k \circ \varphi_k^{(4)} = \psi^{(4)} \circ \pi_k \iff \pi_k \circ \varphi_k^{(4)} = \pi_k$$

where we used the fact that  $\psi^{(4)} = Id$  (see Proposition 1-(i)). Thus, if  $\mathbf{x} \in S_{(p,q)}^k$  one has  $\pi_k(\mathbf{x}) = (p, q)$ , and again from the last equivalence,

$$\pi_k \circ \varphi_k^{(4)}(\mathbf{x}) = \pi_k(\mathbf{x}) = (p, q),$$

meaning that  $\varphi_k^{(4)}(\mathbf{x}) \in S_{(p,q)}^k$ .

Note that the components of  $\pi_k$  are Casimirs of  $P$ , therefore the fact that the Casimirs are first integrals of  $\varphi_k^{(4)}$  is just a consequence of the identity above.  $\square$

### 3 Restrictions to symplectic leaves and the group of symplectic birational maps of the plane

The symplectic leaves  $S_{(p,q)}^k$  defined by (11) are two dimensional semi-algebraic sets invariant under  $\varphi_k^{(4)}$ . However, there are symplectic leaves which are invariant under a lower order iterate of  $\varphi_k$ . Indeed, from (4) and (8) one has

$$\pi_k \circ \varphi_k^{(n)} = \psi^{(n)} \circ \pi_k, \quad n \in \mathbb{Z}_+$$

from which it follows that the symplectic leaves  $S_{(p,q)}^k = \pi_k^{-1}(p, q)$  are invariant under  $\varphi_k^{(n)}$  if and only if  $(p, q)$  is an  $n$ -periodic point of  $\psi$ .

A point fixed by the  $n$ th iterate of a function and not fixed by any other lower order iterate will be called a point of *minimal period*  $n$ .

In  $\mathbb{R}_+^2$ , the map  $\psi$  has a unique fixed point  $(1, 1)$  and any other point is periodic of minimal period 4. So one has the following invariance of the symplectic leaves:

- $S_{(1,1)}^k$  is invariant under  $\varphi_k$ ;

- for any  $(p, q) \neq (1, 1)$ ,  $S_{(p,q)}^k$  is invariant under  $\varphi_k^{(4)}$  and not invariant under any lower order iterate of  $\varphi_k$ .

The expressions of the restriction of  $\varphi_k$  to  $S_{(1,1)}^k$  and of the restriction of  $\varphi_k^{(4)}$  to  $S_{(p,q)}^k$  are given in the following proposition.

**Proposition 2.** *Let  $\varphi_k$  be the maps (1) and  $S_{(p,q)}^k$  the 2-dimensional symplectic leaves defined by (11). Then,*

1.  $S_{(1,1)}^k$  is invariant under  $\varphi_k$  and in the coordinates  $(x_1, x_2)$ , the restriction  $\bar{\varphi}_k = \varphi_k|_{S_{(1,1)}^k}$  is given by:

$$\bar{\varphi}_k(x_1, x_2) = \left( x_2, 2 \frac{x_2^k}{x_1} \right). \quad (13)$$

2. if  $(p, q) \neq (1, 1)$  the symplectic leaves  $S_{(p,q)}^k$  are invariant under  $\varphi_k^{(4)}$  and the restriction  $\tilde{\varphi}_k = \varphi_k^{(4)}|_{S_{(p,q)}^k}$  is given, in the coordinates  $(x_1, x_4)$ , by

$$\tilde{\varphi}_k(x_1, x_4) = \left( \lambda \frac{x_4^{k^2-2}}{x_1}, \lambda^{k^2-1} \frac{x_4^{(k^2-3)(k^2-1)}}{x_1^{k^2-2}} \right), \quad (14)$$

with

$$\lambda = \frac{(1+p^k)^2(1+q^k)^k}{q^2 p^k}. \quad (15)$$

*Proof.* As seen previously the invariance properties of the symplectic leaves follows from the type of periodic points of the map  $\psi$  in (9). Straightforward computations lead to the expressions of the restricted map in (13). To obtain  $\tilde{\varphi}_k$ , the computation of  $\varphi_k^{(4)}(x_1, x_2, x_3, x_4) = (u_1, u_2, u_3, u_4)$  gives

$$\begin{aligned} u_1 &= l(\mathbf{x}) \frac{x_4^{k^2-2}}{x_1}, & u_2 &= l^k(\mathbf{x}) \frac{x_2 x_4^{k^3-3k}}{x_1^k}, \\ u_3 &= l^k(\mathbf{x}) \frac{x_3 x_4^{k^3-3k}}{x_1^k}, & u_4 &= l^{k^2-1}(\mathbf{x}) \frac{x_4^{(k^2-3)(k^2-1)}}{x_1^{k^2-2}}, \end{aligned}$$

where

$$l(\mathbf{x}) = \frac{(x_1^k x_4^k + (x_2^k + x_3^k)^k)^k}{x_1^{k^2-2} x_2^k x_3^k x_4^{k^2-2}}. \quad (16)$$

It is easy to see that the function  $l$  is constant on each  $S_{(p,q)}^k$  and given by

$$\lambda = l(\mathbf{x})|_{S_{(p,q)}^k} = \frac{(1+p^k)^2(1+q^k)^k}{q^2 p^k}.$$

This leads directly to the expression of  $\tilde{\varphi}_k$  in the coordinates  $(x_1, x_4)$ .  $\square$

**Remark 3.** In the above proposition, the different choice of coordinates for  $S_{(1,1)}^k$  and for  $S_{(p,q)}^k$  has no particular meaning other than leading in each case to simpler expressions of the restricted maps.

Using the expressions of the restricted maps (13)-(14) one easily verifies that they are maps of the plane preserving the symplectic form

$$\omega = \frac{1}{xy} dx \wedge dy. \quad (17)$$

The fact that these maps are symplectic is not a coincidence since: (i) the symplectic structure on a symplectic leaf  $S$  is the nondegenerate Poisson structure induced from  $P$ , meaning that the inclusion  $i : S \rightarrow \mathbb{R}_+^4$  is a Poisson map; (ii)  $\varphi_k$  is a Poisson map, and so is any iterate  $\varphi_k^{(n)}$ ; (iii) the restricted maps are just the composition  $\varphi_k^{(n)}|_S = \varphi_k^{(n)} \circ i$  and so they are Poisson maps preserving the symplectic structure on  $S$  induced from  $P$ .

The restricted maps in (13)-(14) belong to the group of birational maps preserving the symplectic form (17). Namely, the group of maps of the form

$$f(x, y) = (\alpha x^a y^b, \beta x^c y^d),$$

with  $\alpha$  and  $\beta$  nonzero constants and  $a, b, c, d$  integers satisfying  $ad - bc = 1$ .

Using algebraic geometry techniques, it was proved by Blanc in [1] that the group of birational transformations of  $\mathbb{C}^2$  preserving the symplectic form (17) is generated by  $SL(2, \mathbb{Z})$ , the complex torus  $(\mathbb{C}^*)^2$  and the globally 5-periodic (Lyness) map  $(x, y) \mapsto (y, \frac{1+y}{x})$ . Here we consider this group restricted to  $\mathbb{R}_+^2$  and we will denote it by  $\Gamma$ .

Let  $\Gamma$  be the group of maps  $f : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+^2$  defined by

$$f(x, y) = (\alpha x^a y^b, \beta x^c y^d), \quad \alpha, \beta \in \mathbb{R}_+, \quad ad - bc = 1, \quad a, b, c, d \in \mathbb{Z}. \quad (18)$$

Using logarithmic coordinates, we can show that  $\Gamma$  is isomorphic to the semidirect product  $SL(2, \mathbb{Z}) \ltimes \mathbb{R}^2$ . Namely, considering the map  $i : \mathbb{R}_+^2 \rightarrow \mathbb{R}^2$  given by

$$i(x, y) = (\log x, \log y), \quad (19)$$

this map conjugates  $f \in \Gamma$  to the affine map in  $\mathbb{R}^2$ :

$$g(u, v) = (au + bv + \log \alpha, cu + dv + \log \beta).$$

Note that  $g$  is the composition of the translation by the vector  $\mathbf{v} = (\log \alpha, \log \beta)$  and an area preserving linear map represented by the  $SL(2, \mathbb{Z})$  matrix

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad ad - bc = 1, \quad a, b, c, d \in \mathbb{Z}. \quad (20)$$



Identifying  $g$  with  $(M, \mathbf{v})$ , the map  $i$  induces an isomorphism between  $\Gamma$  and the semidirect product

$$SL(2, \mathbb{Z}) \ltimes \mathbb{R}^2 = \{(M, \mathbf{v}) : M \in SL(2, \mathbb{Z}), \mathbf{v} \in \mathbb{R}^2\}$$

with group multiplication defined by  $(M, \mathbf{v}) \cdot (N, \mathbf{w}) = (MN, \mathbf{v} + M\mathbf{w})$ .

Let us recall some facts about the special linear group,  $SL(2, \mathbb{Z})$ , over the integers. The elements of  $SL(2, \mathbb{Z})$  are classified into elliptic, parabolic and hyperbolic according to the values of the trace of the matrix  $M \in SL(2, \mathbb{Z})$ . Namely,

- If  $|tr(M)| < 2$ , then  $M$  is called elliptic, and is conjugate to a rotation.
- If  $|tr(M)| = 2$ , then  $M$  is called parabolic, and is a shear map.
- If  $|tr(M)| > 2$ , then  $M$  is called hyperbolic.

In order to better identify the type of periodic points of the restricted maps in Proposition 2 we deduce in the next proposition a normal form for all the maps in  $\Gamma$  except the maps

$$f_{(\alpha, \beta)}^{\pm}(x, y) = (\alpha x^{\pm 1}, \beta y^{\pm 1}).$$

**Proposition 3.** *Let  $f : \mathbb{R}_+^2 \rightarrow \mathbb{R}^2$ , defined by*

$$f(x, y) = (\alpha x^a y^b, \beta x^c y^d), \quad ad - bc = 1 \quad \alpha, \beta \neq 0$$

*be an element of  $\Gamma$  with  $b^2 + c^2 \neq 0$ . Then  $f$  is conjugate to the map:*

1.  $f_{a+d}(x, y) = (y, \frac{y^{a+d}}{x})$ , if  $a + d \neq 2$ ;
2.  $f_{2, \xi}(x, y) = (y, \xi \frac{y^2}{x})$ , if  $a + d = 2$ , where

$$\xi = \begin{cases} \frac{\alpha^c}{\beta^{\alpha-1}}, & \text{if } c \neq 0 \\ \beta^b, & \text{if } c = 0. \end{cases} \quad (21)$$

*Proof.* If  $c \neq 0$ , considering the homeomorphism  $\pi$  given by

$$\pi(x, y) = (y^a x^{-c}, \beta^a \alpha^{-c} y),$$

it is easy to check that  $\pi \circ f = g \circ \pi$ , where  $g$  is the map

$$g(x, y) = (y, K \frac{y^{a+d}}{x}) \quad \text{with} \quad K = \beta(\beta^a \alpha^{-c})^{1-(a+d)}.$$

If  $a + d = 2$  the map  $g$  is the map  $f_{2, \xi}$  with  $\xi = \frac{\alpha^c}{\beta^{\alpha-1}}$ . If  $a + d \neq 2$ , taking the following map  $\Pi$

$$\Pi(x, y) = K^{\frac{1}{a+d-2}}(x, y),$$

we have  $\Pi \circ g = f_{a+d} \circ \Pi$ , that is  $\Pi \circ \pi \circ f = f_{a+d} \circ \Pi \circ \pi$ .

If  $c = 0$ , the hypothesis  $b^2 + c^2 \neq 0$  implies that  $b \neq 0$ . Considering the involution  $\sigma(x, y) = (y, x)$ , which interchanges  $c$  and  $b$ , the problem reduces to the previous cases. In fact,  $\sigma \circ f \circ \sigma = (\beta x^d, \alpha x^b y^a)$  is conjugate to  $f_{a+d}$  if  $a + d \neq 2$  and to  $f_{2, \xi}$  with  $\xi = \frac{\beta^b}{\alpha^{d-1}} = \beta^b$  if  $a + d = 2$ .  $\square$

**Remark 4.** It is worth noting that the conjugacies in the proof of the above proposition belong to a group  $G$  which is isomorphic to  $GL(2, \mathbb{Z}) \rtimes \mathbb{R}^2$ . The result in the proposition may be rephrased as follows. Up to conjugation in  $G$ , the elements  $(M, \mathbf{v}) \in SL(2, \mathbb{Z}) \rtimes \mathbb{R}^2$ , with  $M \neq \pm I$ , are parametrized: (a) by the trace of  $M$  if  $\text{tr } M \neq 2$ ; (b) by a real parameter  $\xi$  depending on  $M$  and  $\mathbf{v}$  through the expression (21), in the case of  $\text{tr } M = 2$ .

As a consequence of the proof of Proposition 3 the restricted maps (13) and (14) are conjugate to the normal forms given in the following corollary.

**Corollary 2.** *Let  $k \in \mathbb{Z}_+$  and  $\lambda$  be a nonzero real number. Consider the maps*

$$\bar{\varphi}_k(x, y) = \left( y, 2 \frac{y^k}{x} \right), \quad \tilde{\varphi}_k(x, y) = \left( \lambda \frac{y^{k^2-2}}{x}, \lambda^{k^2-1} \frac{y^{(k^2-3)(k^2-1)}}{x^{k^2-2}} \right).$$

1. *If  $k = 2$ , then*

- i)  $\bar{\varphi}_2$  is already in normal form:  $\bar{\varphi}_2 = f_{2,2}$ ;
- ii)  $\tilde{\pi}_2 \circ \tilde{\varphi}_2 = f_{2, \lambda^4} \circ \tilde{\pi}_2$  with

$$\tilde{\pi}_2(x, y) = \left( \frac{x^2}{y}, \frac{y}{\lambda} \right), \quad f_{2, \lambda^4}(x, y) = \left( y, \lambda^4 \frac{y^2}{x} \right). \quad (22)$$

2. *If  $k \neq 2$  then,*

- i)  $\bar{\pi}_k \circ \bar{\varphi}_k = f_k \circ \bar{\pi}_k$  with

$$\bar{\pi}_k(x, y) = 2^{\frac{1}{k-2}}(x, y), \quad f_k(x, y) = \left( y, \frac{y^k}{x} \right); \quad (23)$$

- ii)  $\tilde{\pi}_k \circ \tilde{\varphi}_k = f_{(k^2-2)^2-2} \circ \tilde{\pi}_k$  with

$$\tilde{\pi}_k(x, y) = \lambda^{\frac{1}{k^2-4}} \left( \lambda \frac{x^{k^2-2}}{y}, y \right),$$

$$f_{(k^2-2)^2-2}(x, y) = \left( y, \frac{y^{(k^2-2)^2-2}}{x} \right). \quad (24)$$

*Proof.* Note that both maps  $\bar{\varphi}_k$  and  $\tilde{\varphi}_k$  verify the hypotheses of Proposition 3 with  $c \neq 0$ , for any  $k$ . Furthermore,  $a + d = k$  for  $\bar{\varphi}_k$  and  $a + d = (k^2 - 2)^2 - 2$  for  $\tilde{\varphi}_k$ . For both maps  $a + d = 2$  if and only if  $k = 2$ . The result then follows from the proof of Proposition 3.  $\square$

**Remark 5.** We remark that from the proof of the above corollary the restricted maps  $\bar{\varphi}_k$  and  $\tilde{\varphi}_k$  are conjugate to  $SL(2, \mathbb{Z})$  maps except in the case  $k = 2$ . Moreover: (a)  $|a + d| = 1$  if and only if  $k = 1$ , so that  $\bar{\varphi}_1$  and  $\tilde{\varphi}_1$  are conjugate to elliptic  $SL(2, \mathbb{Z})$  maps; (b)  $|a + d| > 2$  if and only if  $k \geq 3$ , and so  $\bar{\varphi}_k$  and  $\tilde{\varphi}_k$  are conjugate to hyperbolic  $SL(2, \mathbb{Z})$  maps for  $k \geq 3$ .

For future reference, we now mention the form of an iterate of order  $n$  of the maps  $f_{2,\xi}(x, y) = (y, \xi \frac{y^2}{x})$ , given in Proposition 3-2. This expression can be computed by applying Lemma 1 in [3] or by considering the conjugate affine map

$$g_{2,\xi}(u, v) = (v, -u + 2v + \log \xi), \quad \xi \in \mathbb{R}_+.$$

This map can be identified with the  $SL(3, \mathbb{R})$  matrix

$$X = \left[ \begin{array}{cc|c} 0 & 1 & 0 \\ -1 & 2 & \log \xi \\ 0 & 0 & 1 \end{array} \right],$$

and so, computing the  $n$ th power of  $X$  we arrive at the expression of  $g_{2,\xi}^{(n)}$  from which we obtain:

$$f_{2,\xi}^{(n)}(x, y) = \xi^{\frac{n(n-1)}{2}} \frac{y^n}{x^n} (x, \xi^n y), \quad n \geq 0. \quad (25)$$

## 4 Periodic Points of the maps $\varphi_k$

The existence of periodic points for the maps of the family  $\varphi_k$  given by (1) is obtained from the periodic points of the maps restricted to the symplectic leaves, namely the maps  $\bar{\varphi}_k$  and  $\tilde{\varphi}_k$  in Proposition 2. In turn, the existence of periodic points of these restricted maps rely on the results of the previous section for the group  $\Gamma$ .

### 4.1 Periodic points of the restricted maps

In this subsection we describe the type of periodic points of the restricted maps according to the values of the parameter  $k$ .

**Proposition 4.** *Let  $S_{(p,q)}^k \subseteq \mathbb{R}_+^2$  be the symplectic leaves defined by (11) and  $\bar{\varphi}_k, \tilde{\varphi}_k$  the restrictions*

$$\bar{\varphi}_k = \varphi_k|_{S_{(1,1)}^k}, \quad \tilde{\varphi}_k = \varphi_k^{(4)}|_{S_{(p,q)}^k}, \quad (p, q) \neq (1, 1)$$

*given by (13)-(15). The periodic points of these maps are as follows.*

1. For  $k = 1$ :

(i) in  $S_{(1,1)}^1$ , the map  $\bar{\varphi}_1$  has a unique fixed point  $(2, 2)$  and any other point is periodic of minimal period 6.

- (ii) in  $S_{(p,q)}^1$ ,  $(p, q) \neq (1, 1)$ , the map  $\tilde{\varphi}_1$  has a unique fixed point  $(\lambda^{1/3}, \lambda^{1/3})$  and any other point is periodic of minimal period 3.
2. For  $k = 2$ , the maps  $\bar{\varphi}_2$  and  $\tilde{\varphi}_2$  have no periodic points.
3. For  $k \geq 3$ :
- (i) in  $S_{(1,1)}^k$ , the map  $\bar{\varphi}_k$  has a unique fixed point  $(2^{\frac{1}{2-k}}, 2^{\frac{1}{2-k}})$  and no other periodic points.
- (ii) in  $S_{(p,q)}^k$ ,  $(p, q) \neq (1, 1)$ , the map  $\tilde{\varphi}_k$  has a unique fixed point  $(\lambda^{\frac{1}{4-k^2}}, \lambda^{\frac{1}{4-k^2}})$  and no other periodic points.

*Proof.* 1. From Corollary 2-2, both maps  $\bar{\varphi}_1$  and  $\tilde{\varphi}_1$  are conjugate to elliptic  $SL(2, \mathbb{Z})$  maps. The map  $\bar{\varphi}_1$  is conjugate to  $f_1(x, y) = (y, \frac{y}{x})$  which is a globally 6-periodic map with a unique fixed point  $(1, 1)$ . So,  $\bar{\varphi}_1$  is globally 6-periodic with a unique fixed point  $(2, 2)$ . Analogously, the map  $\tilde{\varphi}_1$  is conjugate to  $f_{-1}(x, y) = (y, \frac{1}{xy})$  which is a globally 3-periodic map with a unique fixed point  $(1, 1)$ . Hence,  $\tilde{\varphi}_1$  is globally 3-periodic with a unique fixed point  $(\lambda^{1/3}, \lambda^{1/3})$ .

2. First note that by Corollary 2-1 the maps  $\bar{\varphi}_2$  and  $\tilde{\varphi}_2$  are conjugate to  $f_{2,2}$  and  $f_{2,\lambda^4}$  respectively, with

$$\bar{\varphi}_2(x, y) = f_{2,2}(x, y) = \left( y, 2\frac{y^2}{x} \right), \quad \tilde{\varphi}_2 = \tilde{\pi}_2^{-1} \circ f_{2,\lambda^4} \circ \tilde{\pi}_2.$$

Using (25) with  $\xi = 2$  one has

$$\bar{\varphi}_2^{(n)}(x, y) = 2^{\frac{n(n-1)}{2}} \left( \frac{y^n}{x^{n-1}}, \frac{2^n y^{n+1}}{x^n} \right),$$

from which the result follows.

By noting that  $\tilde{\varphi}_2^{(n)} = \tilde{\pi}_2^{-1} \circ f_{2,\lambda^4}^{(n)} \circ \tilde{\pi}_2$ , applying again (25) to obtain  $f_{2,\lambda^4}^{(n)}$  and taking into account that

$$\tilde{\pi}_2(x, y) = \left( \frac{x^2}{y}, \frac{y}{\lambda} \right), \quad \tilde{\pi}_2^{-1}(x, y) = \left( \sqrt{\lambda xy}, \lambda y \right)$$

and that  $\lambda = \left(p + \frac{1}{p}\right)^2 \left(q + \frac{1}{q}\right)^2 > 1$ , it is easy to see from the expression of  $\tilde{\varphi}_2^{(n)}$  that  $\tilde{\varphi}_2$  has no periodic points.

3. For  $k \geq 3$ , from Corollary 2-2, both  $\bar{\varphi}_k$  and  $\tilde{\varphi}_k$  are conjugate, respectively, to the  $SL(2, \mathbb{Z})$  hyperbolic maps

$$f_k(x, y) = \left( y, \frac{y^k}{x} \right) \quad \text{and} \quad f_{(k^2-2)^2-2}(x, y) = \left( y, \frac{y^{(k^2-2)^2-2}}{x} \right).$$

Note that  $(k^2 - 2)^2 - 2 > 2$  when  $k \geq 3$ . Therefore, as  $f_k$  and  $f_{(k^2-2)^2-2}$  have no periodic points other than the fixed point, the same happens for the maps  $\bar{\varphi}_k$  and  $\tilde{\varphi}_k$ . The computation of the fixed points of  $\bar{\varphi}_k$  and  $\tilde{\varphi}_k$  gives the result.  $\square$

## 4.2 Periodic points of $\varphi_k$

Finally, we address the problem of describing the main dynamical features of the maps of the family (1) defined in  $\mathbb{R}_+^4$ . Recall that, by Theorem 1,  $\mathbb{R}_+^4$  is foliated by 2-dimensional symplectic leaves  $S_{(p,q)}^k$  of  $P$  (with  $P$  as in (2)), all of them invariant under the fourth iterate  $\varphi_k^{(4)}$  and with the leaf  $S_{(1,1)}^k$  invariant under  $\varphi_k$ . In particular, this means that each orbit of  $\varphi_k$  is either entirely contained in  $S_{(1,1)}^k$  or jumps between four pairwise disjoint symplectic leaves

$$S_{(p,q)}^k \rightarrow S_{(q,p^{-1})}^k \rightarrow S_{(p^{-1},q^{-1})}^k \rightarrow S_{(q^{-1},p)}^k$$

all of them invariant under  $\varphi_k^{(4)}$ . In fact, the leaves  $S_{(p,q)}^k$  are the fibres of  $\pi_k$  and by (12) one has  $\pi_k \circ \varphi_k = \psi \circ \pi_k$  with  $\psi(x, y) = (y, \frac{1}{x})$ . So,

$$S_{(p,q)}^k \xrightarrow{\varphi_k} S_{\psi(p,q)}^k.$$

In Theorem 3 and Theorem 4 below we will characterize the periodic points of  $\varphi_k$  in the cases  $k = 1$  and  $k \geq 3$ , respectively. The case  $k = 2$  will not be explicitly stated since it is easy to see that the map  $\varphi_2$  has no periodic points and its dynamics is described in detail in our work [3, Theorem 3].

**Theorem 3.** *Let  $\varphi_1 : \mathbb{R}_+^4 \rightarrow \mathbb{R}_+^4$  be the map of the family (1):*

$$\varphi_1(x_1, x_2, x_3, x_4) = \left( x_3, x_4, \frac{x_2 + x_3}{x_1}, \frac{x_1 x_4 + x_2 + x_3}{x_1 x_2} \right)$$

and consider the symplectic foliation of  $\mathbb{R}_+^4$  defined by (11) with

$$S_{(p,q)}^1 = \{ \mathbf{x} \in \mathbb{R}_+^4 : x_3 = px_2, qx_1x_4 = (1+p)x_2 \}.$$

1. *The map  $\varphi_1$  is globally 12-periodic.*
2. *In the symplectic leaf  $S_{(1,1)}^1$  there is exactly one fixed point  $F = (2, 2, 2, 2)$  of  $\varphi_1$  and any other point of  $S_{(1,1)}^1$  is periodic of minimal period 6.*
3. *Each symplectic leaf  $S_{(p,q)}^1$ , with  $(p, q) \neq (1, 1)$ , contains a 2-dimensional semi-algebraic set*

$$V = \{ (x_1, x_2, x_3, x_4) \in \mathbb{R}_+^4 : x_4 = x_1, x_1x_2x_3 = x_1^2 + x_2 + x_3 \},$$

*of points of minimal period 4 and any other point in  $S_{(p,q)}^1 \setminus V$  is periodic with minimal period 12.*

*Proof.* By Proposition 4-1, the restriction  $\bar{\varphi}_1$  of  $\varphi_1$  to  $S_{(1,1)}^1$  is globally 6-periodic and the restriction  $\tilde{\varphi}_1$  of  $\varphi_1^{(4)}$  to any  $S_{(p,q)}^1$  is globally 3-periodic. Hence  $\varphi_1$  is globally 12-periodic. Moreover, all the points in  $S_{(1,1)}^1$  have minimal period 6 except the point  $F = (2, 2, 2, 2)$  which is fixed. Also, any point belonging to  $S_{(p,q)}^1$  is either a fixed point of  $\varphi_1^{(4)}$  or a periodic point of  $\varphi_1^{(4)}$  with minimal period 3.

To compute the fixed points of  $\varphi_1^{(4)}$ , which correspond to periodic points of  $\varphi_1$  with minimal period 4, we note that by Proposition 4-1 and Proposition 2 these are points  $\mathbf{x} \in S_{(p,q)}^1$  whose coordinates  $x_1$  and  $x_4$  satisfy  $x_1 = x_4 = \lambda^{1/3}$ , for  $\lambda$  given by (15) (with  $k = 1$ ). On the other hand, the constant  $\lambda$  is the value of the restriction to  $S_{(p,q)}^1$  of the function  $l(\mathbf{x})$  given in (16). To obtain the set  $V$  it is enough to eliminate  $\lambda$  from these relations, that is from

$$x_1 = x_4 = \lambda^{1/3}, \quad \lambda = \frac{x_1 x_4 + x_2 + x_3}{x_1^{-1} x_2 x_3 x_4^{-1}}.$$

Finally, the remaining points are periodic points of  $\varphi_1^{(4)}$  with minimal period 3, and therefore they are periodic points of  $\varphi_1$  with minimal period 12.  $\square$

**Theorem 4.** For each integer  $k \geq 3$ , let  $\varphi_k : \mathbb{R}_+^4 \rightarrow \mathbb{R}_+^4$  be the map

$$\varphi_k(x_1, x_2, x_3, x_4) = \left( x_3, x_4, \frac{x_2^k + x_3^k}{x_1}, \frac{x_1^k x_4^k + (x_2^k + x_3^k)^k}{x_1^k x_2} \right)$$

and the symplectic leaves

$$S_{(p,q)}^k = \{ \mathbf{x} \in \mathbb{R}_+^4 : x_3 = p x_2, q x_1 x_4 = (1 + p^k) x_2^k \}.$$

Then,  $\varphi_k$  has no periodic points other than:

1. a unique fixed point  $F = (2^{\frac{1}{2-k}}, 2^{\frac{1}{2-k}}, 2^{\frac{1}{2-k}}, 2^{\frac{1}{2-k}})$  belonging to  $S_{(1,1)}^k$ .
2. each leaf  $S_{(p,q)}^k$ , with  $(p, q) \neq (1, 1)$ , contains a semi-algebraic set  $V \subset S_{(p,q)}^k$  of periodic points of  $\varphi_k$  with minimal period 4 given by

$$V = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}_+^4 : x_4 = x_1, x_1^k x_2 x_3 = x_1^{2k} + (x_2^k + x_3^k)^k \right\}.$$

*Proof.* The proof follows the same lines of the proof of the previous theorem by considering the periodic points of the restriction  $\bar{\varphi}_k$  of  $\varphi_k$  to  $S_{(1,1)}^k$  and the restrictions  $\tilde{\varphi}_k$  of  $\varphi_k^{(4)}$  to each  $S_{(p,q)}^k$  with  $(p, q) \neq (1, 1)$ . By Proposition 2 the restriction  $\bar{\varphi}_k$  is given in the coordinates  $(x_1, x_2)$  by (13) and  $\tilde{\varphi}_k$  is given in the coordinates  $(x_1, x_4)$  by (14).

By Proposition 4-3-(i),  $\bar{\varphi}_k$  has a unique fixed point  $(x_1, x_2) = (2^{\frac{1}{2-k}}, 2^{\frac{1}{2-k}})$  and no other periodic points. This fixed point corresponds to the fixed point  $F = (2^{\frac{1}{2-k}}, 2^{\frac{1}{2-k}}, 2^{\frac{1}{2-k}}, 2^{\frac{1}{2-k}})$  of  $\varphi_k$ . Also, by Proposition 4-3-(ii), for each

$(p, q) \neq (1, 1)$  the restriction  $\tilde{\varphi}_k$  of  $\varphi_k^{(4)}$  to  $S_{(p,q)}^k$  has a unique fixed point  $(x_1, x_4) = (\lambda^{\frac{1}{4-k^2}}, \lambda^{\frac{1}{4-k^2}})$  and no other periodic points. Each of these fixed points corresponds to a periodic point of  $\varphi_k$  with minimal period 4.

The full set of these 4-periodic points is a 2-dimensional set  $V \subset S_{(p,q)}^k$ . Like in the proof of the previous theorem, the explicit form of  $V$  is easily obtained from the fact that the fixed point of  $\tilde{\varphi}_k$  satisfies  $x_1 = x_4 = \lambda^{\frac{1}{4-k^2}}$  and from the fact that  $\lambda$  is the value of the restriction to  $S_{(p,q)}^k$  of the function  $l(\mathbf{x})$  given in (16).  $\square$

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