1 Polinómio característico:

$$p(\lambda) = \begin{vmatrix} (2-\lambda) & 1 & 1 \\ 2 & 3-\lambda & 4 \\ -1 & -1 & -(2+\lambda) \end{vmatrix} \begin{aligned}[t] &= (2-\lambda)\big((\lambda-3)(\lambda+2)+4\big) - \big(-2(\lambda+2)+4\big) + \big(-2+3-\lambda\big) \\ &= (2-\lambda)(\lambda^2-\lambda-2) + 2\lambda + 1 - \lambda \\ &= (2-\lambda)(\lambda-2)(\lambda+1) + \lambda + 1 \\ &= -(\lambda+1)\big((\lambda-2)^2-1\big) = -(\lambda+1)(\lambda-1)(\lambda-3) \end{aligned}$$

Valores próprios: $\lambda_1 = -1,\ \lambda_2 = 1,\ \lambda_3 = 3.$
Vectores próprios:

$$(A+I)u_1 = 0 \quad \Leftrightarrow \quad \begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 4 \\ -1 & -1 & -1 \end{bmatrix} u_1 = 0 \quad \Rightarrow u_1 = \alpha(0,1,-1), \quad \alpha \in \mathbb{R}; \quad \dim E(\lambda_1) = 1$$

$$(A-I)u_2 = 0 \quad \Leftrightarrow \quad \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 4 \\ -1 & -1 & -3 \end{bmatrix} u_2 = 0 \quad \Rightarrow u_2 = \beta(1,-1,0), \quad \beta \in \mathbb{R}; \quad \dim E(\lambda_2) = 1$$

$$(A-3I)u_3 = 0 \quad \Leftrightarrow \quad \begin{bmatrix} -1 & 1 & 1 \\ 2 & 0 & 4 \\ -1 & -1 & -5 \end{bmatrix} u_3 = 0 \quad \Rightarrow u_3 = \gamma(2,3,-1), \quad \gamma \in \mathbb{R}; \quad \dim E(\lambda_3) = 1$$

2 Polinómio característico:

$$p(\lambda) = \begin{vmatrix} (2-\lambda) & 1 & 1 \\ 2 & 3-\lambda & 2 \\ 3 & 3 & -(4-\lambda) \end{vmatrix} \begin{aligned}[t] &= (2-\lambda)\big((\lambda-3)(\lambda-4)-6\big) - \big(2(4-\lambda)-6\big) + \big(6-3(3-\lambda)\big) \\ &= (2-\lambda)(\lambda^2-7\lambda+6) + 2(\lambda-1) + 3(\lambda-1) \\ &= (2-\lambda)(\lambda-1)(\lambda-6) + 5(\lambda-1) \\ &= -(\lambda-1)\big((\lambda-2)(\lambda-6)-5\big) \\ &= -(\lambda-1)(\lambda^2-8\lambda+7) = -(\lambda-1)^2(\lambda-7) \end{aligned}$$

Valores próprios: $\lambda_1 = \lambda_2 = 1,\ \lambda_3 = 7.$
Vectores próprios:

$$(A-I)u = 0 \quad \Leftrightarrow \quad \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} u = 0 \quad \Rightarrow \begin{array}{l} \text{2 soluções linearmente independentes, por exemplo:} \\ u_1 = \alpha(1,0,-1), \quad \alpha \in \mathbb{R}, \\ u_2 = \beta(0,1,-1), \quad \beta \in \mathbb{R}, \quad \dim E(\lambda_1) = 2 \end{array} ;$$

$$(A-7I)u_3 = 0 \quad \Leftrightarrow \quad \begin{bmatrix} -5 & 1 & 1 \\ 2 & -4 & 2 \\ 3 & 3 & -3 \end{bmatrix} u_3 = 0 \quad \Rightarrow u_3 = \gamma(1,2,3), \quad \gamma \in \mathbb{R}; \quad \dim E(\lambda_3) = 1$$

$\boxed{3}$ Polinómio característico:

$$p(\lambda)=\begin{vmatrix}(2-\lambda) & -1 & 1\\ 0 & 3-\lambda & -1\\ 2 & 1 & 3-\lambda\end{vmatrix} = (2-\lambda)\big((3-\lambda)^2+1\big)+2\big(1+\lambda-3\big))$$

$$= (2-\lambda)(\lambda^2-6\lambda+10)+2(\lambda-2)$$

$$= -(\lambda-2)(\lambda^2-6\lambda+8)$$

$$= -(\lambda-2)^2(\lambda-4)$$

Valores próprios: $\lambda_1=\lambda_2=2,\ \lambda_3=4.$
Vectores próprios:

$$(A{-}I)u_1=0 \quad\Leftrightarrow\quad \begin{bmatrix}0 & -1 & 1\\ 0 & 1 & -1\\ 2 & 1 & 1\end{bmatrix}u_1=0 \quad\Rightarrow u_1=\alpha(1,-1,-1),\quad \alpha\in\mathbb{R};\quad \dim E(\lambda_1)=1$$

$$(A{-}4I)u_3=0 \quad\Leftrightarrow\quad \begin{bmatrix}-2 & -1 & 1\\ 0 & -1 & -1\\ 2 & 1 & -1\end{bmatrix}u_3=0 \quad\Rightarrow u_3=\gamma(-1,1,-1),\quad \gamma\in\mathbb{R};\quad \dim E(\lambda_3)=1$$

$\boxed{4}$ Polinómio característico:

$$p(\lambda)=\begin{vmatrix}(4-\lambda) & 1 & -1\\ 0 & 3-\lambda & 1\\ 2 & 1 & 5-\lambda\end{vmatrix} = (4-\lambda)\big((3-\lambda)(5-\lambda)-1\big)+2\big(1+3-\lambda\big))$$

$$= (4-\lambda)(\lambda^2-8\lambda+14)-2(\lambda-4)$$

$$= -(\lambda-4)(\lambda^2-8\lambda+16)$$

$$= -(\lambda-4)^3$$

Valores próprios: $\lambda_1=\lambda_2=\lambda_3=4.$
Vectores próprios:

$$(A{-}4I)u_1=0 \quad\Leftrightarrow\quad \begin{bmatrix}0 & 1 & -1\\ 0 & -1 & 1\\ 2 & 1 & 1\end{bmatrix}u_1=0 \quad\Rightarrow u_1=\alpha(1,-1,-1),\quad \alpha\in\mathbb{R};\quad \dim E(\lambda_1)=1$$

Nota: No cálculo dos determinantes usou-se sempre a fórmula de Laplace. Nos casos $\boxed{1}$ e $\boxed{2}$ expandiu-se o determinante segundo a primeira linha da matriz, enquanto que nos casos $\boxed{3}$ e $\boxed{4}$ se usou a primeira coluna para efectuar a expansão.